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GRAMMAR SCHOOL ALGEBRA

BY

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E. V. Hill

PREFACE.

THE desirable results to be attained in the study of Algebra are the thought of letters as general symbols of number, familiarity with the laws of the algebraic notation, facility and accuracy in performing algebraic processes, skill in the use of the equation as a means of mathematical investigation, and the development of the reasoning powers. In the attainment of these results, the teacher is the important factor, but the character of the book used determines in a greater or less degree the character of the work done and the effects upon the student.

These considerations have been the motives for the preparation of this work, and it is confidently believed that the use of it will lead to broader views of numbers and their properties, and produce increasing mental power. It is designed for use in high schools and academies, and advanced classes in grammar schools. A distinguishing feature of the book is its arrangement. The definitions, demonstration of principles, derivation of rules, model solutions, and illustrations occupy the last half of the book, and the exercises and problems the first half. The reasons for this arrangement are evident, and it is believed that the separation of the text from the exercises and problems can be no inconvenience in the legitimate use of either. The text is complete in the clearness and conciseness of definitions, thorough demonstration of principles, careful derivation of rules, and the abundance of illustrations and model

solutions. Frequent notes to the teacher will be found in the first half of the book. The numbers in these notes refer to articles in the text, and the notes indicate that the following exercise contains new work, for which students must be prepared.

The other prominent features of the work are the frequent exercises in algebraic expression, the unusual number and variety of examples and problems and the careful grading of the same, the early introduction of the equation and its use in the solution of problems, the thorough treatment of factoring, the completeness of the work on involution and evolution, the discussions and exercises on the signification of exponents, the character and amount of work in radicals, and the emphasis on the solution of complete quadratics by the method of factoring.

The author desires to acknowledge his indebtedness to A. B. Davis, A.M., Principal of High School, Mount Vernon, N.Y.; Lyman A. Best, A.M., Principal of School No. 13, Brooklyn, N.Y.; and J. Frank Shields, B.S., Professor of Mathematics, Adelphi College, Brooklyn, N.Y.; all of whom have critically read the MS. and made valuable suggestions concerning various features of the work.

The work is submitted to the profession with the hope that teachers and students may use it with the greatest pleasure and profit. This is the highest reward the author can desire.

GEORGE E. ATWOOD.

CONTENTS.

ALGEBRA.

INTRODUCTION.

1. Algebra is not altogether unlike arithmetic. Both treat of numbers, but the method of representing numbers is not the same in both. In arithmetic, numbers are represented by ten characters, which are called *figures*. Each figure, when standing alone, represents a definite and fixed value, or number of things. The figure 4 standing alone always represents four things; the figure 7, seven things; the figure 9, nine things.

2. In algebra, numbers are represented by letters, by figures, or by a combination of figures and letters. The figures used in algebra always represent definite numbers, as in arithmetic, but the letters represent any numbers. The letter x, for example, may represent 5 in one problem, 15 in another, 36 in another, $7\frac{2}{5}$ in another. When figures are written with letters, without any sign between them, the figure indicates how many times the quantity, or number represented by the letter, is taken; and when one letter is written with another, either one of the letters indicates how many times the quantity, or number represented by the other letter, is taken. For example,

Whatever number x represents,

$$3x \text{ means } 3 \text{ times } x,$$
$$5x \text{ means } 5 \text{ times } x,$$
$$ax \text{ means } a \text{ times } x.$$

3. The signs of addition, subtraction, multiplication, and division are used in algebra with the same signification that they have in arithmetic.

EXERCISES IN ALGEBRAIC EXPRESSION.

1. If x represents a certain number, what does $4x$ represent? $2x$? $9x$? ax? bx?

2. If m represents a certain number, what represents 6 times the number? a times the number?

3. Indicate the sum of 8 and 7. Of x and 5. Of a and b. Of a, b, and c.

4. Indicate the difference between x and y, when x is greater than y. When y is greater than x.

5. Indicate the sum of a and b diminished by c. The sum of x and $3x$ diminished by y.

6. Indicate in two ways the product of 5 and a. Of a and x. Of 3, x, and y. Of a, b, and x.

7. A man paid x dollars for a harness and $4x$ dollars for a horse. How much did both cost?

8. If x represents the price of a yard of silk, what does $7x$ represent? $3x$? ax?

9. If a man works for x dollars a day, how much will he earn in nine days? In a days?

10. A man is x years old to-day. How old was he 7 years ago? Sixteen years ago?

11. If a yard of ribbon is worth a cents, how much is a foot worth? Two feet?

12. If you are x years old to-day, how old will you be in 8 years? In twelve years?

13. A man bought a carriage for x dollars and sold it at a loss of y dollars. How much did he get for it?

14. A lady paid a dollars for silk at b dollars a yard. How many yards did she buy?

15. A man is 3 times as old as his son. If the son is x years old, how old is the father?

16. A boy had a dollars. He earned b dollars more, and then spent c dollars. How much had he left?

17. A man sold a horse for x dollars, thereby gaining y dollars. Find the cost of the horse.

18. A boy bought a apples at n cents apiece. How much did he pay for them?

19. I have x dollars. If I pay two debts of a dollars and b dollars, how much shall I have left?

20. A horse cost $5x$ dollars, a harness x dollars, and a carriage $4x$ dollars. Express the cost of all in two ways.

21. A boy has x dimes. How many cents has he? How many dollars has he?

22. A grocer bought a barrels of flour for x dollars. What was the price of the flour per barrel?

23. If the difference between two numbers is 12 and the smaller one is x, what is the larger number?

24. A boy bought x apples at m cents apiece and sold them at n cents. If he gained, what was his gain?

25. If one number is x, and another number is 4 times as great, what is the sum of the numbers?

26. How long will it take a man to walk x miles at the rate of 5 miles an hour?

27. A man was x years old a years ago. How old will he be in b years? How old was he c years ago?

28. If one part of 7 is x, what is the other part? If one part of x is a, what is the other part?

29. A man worked x hours a day for eight days at y cents an hour. With the money earned he bought a coat for a dollars. How much money did he have left?

30. A has x sheep, B has twice as many as A, and C has twice as many as A and B together. What is the value of all their sheep at a dollars a head?

31. A man paid a dollars for b yards of silk. At the same price, how much will c yards cost?

32. A boy bought x oranges at a cents each and sold them at b cents. If he lost, what was his loss?

33. In b years a man will be x years old. How old was he c years ago?

34. If thirty dollars is divided equally among x boys, how much will each boy receive?

35. A has x cows, B has 5 more than A, and C has as many as A and B together. How many have all?

36. How far can a man walk in x hours at the rate of a miles in b hours?

37. A grocer bought a pounds of coffee at x cents a pound and sold it at y cents a pound. If y is greater than x, did he gain or lose, and how much? If x is greater than y, did he gain or lose, and how much?

38. A man bought a pounds of tea at x cents a pound, and paid for it in butter at b cents a pound. How many pounds of butter did he give for the tea?

39. A dealer bought a crates of berries at n cents a quart, and b crates of another kind at m cents a quart. If each crate contained x quarts, how much did they cost?

40. A merchant bought a yards of silk at x dollars a yard. If he sold it all at a profit of y dollars a yard, how much did he receive for it?

41. A speculator bought x acres of land at a dollars an acre. If he sold it all at a loss of b dollars an acre, how much did he receive for it?

4. The *sign of equality* is used in arithmetic to indicate that the numbers between which it is placed are equal.

The expression $9 + 7 = 8 \times 2$ denotes that the sum of 9 and 7 is equal to the product of 8 and 2.

The expression $8 - 5 = 9 \div 3$ denotes that the difference between 8 and 5 is equal to the quotient of 9 divided by 3.

5. In like manner, the sign of equality is used in algebra to indicate that numbers represented wholly or partly by letters are equal.

The expression $a + x = by$ denotes that the sum of the numbers represented by a and x is equal to the product of the numbers represented by b and y.

The expression $ax - 5 = bx + 9$ denotes that the product of a and x diminished by 5 is equal to the product of b and x increased by 9.

6. An Equation is the expression of the equality of two numbers.

$3x = 12.$	$4x = 20.$	$2x - 7 = 9.$
$3x = 15.$	$4x = 32.$	$2x + 2 = 8.$
$3x = 10.$	$4x = 30.$	$2x - 5 = 7.$

Since $3x$, in the above equations, means 3 times x;

$3x$ and 12 are equal only when x represents 4,
$3x$ and 15 are equal only when x represents 5,
$3x$ and 10 are equal only when x represents $3\frac{1}{3}$.

Since $4x$, in the above equations, means 4 times x;

$4x$ and 20 are equal only when x represents 5,
$4x$ and 32 are equal only when x represents 8,
$4x$ and 30 are equal only when x represents $7\frac{1}{2}$.

Since $2x$, in the above equations, means 2 times x;

$2x - 7$ and 9 are equal only when x represents 8,
$2x + 2$ and 8 are equal only when x represents 3,
$2x - 5$ and 7 are equal only when x represents 6.

It is evident from the above that when an equation contains only one letter, as x, the letter used represents some *particular* number.

The number represented by the letter in such an equation is called the *unknown number*.

EXERCISES.

1. If $7x = 28$, what does x represent
2. If $5x = 40$, what does x represent?
3. If $3x = 29$, what does x represent?
4. If $3x + x = 24$, what does x represent?
5. If $2x - x = 17$, what does x represent?
6. If $2x + 3x = 27$, what does x represent?
7. If $7x - 3x = 36$, what does x represent?
8. If $5x - 4x = 15$, what does x represent?
9. If $4x + 3x + x = 32$, what does x represent?
10. If $7x - 3x - x = 36$, what does x represent?
11. If $9x + 4x - x = 60$, what does x represent?
12. If $4x + 2x + 3x = 54$, what does x represent?
13. If $5x + 3x - 7x = 13$, what does x represent?
14. If $9x - 4x - 2x = 30$, what does x represent?
15. If $8x - 3x - 4x = 17$, what does x represent?
16. If $7x - 4x + 5x = 23$, what does x represent?

7. The process of finding the number represented by the letter in an equation is called the *solution of the equation.*

8. Problems in arithmetic or algebra involve an unknown number or numbers, and the process by which the unknown number is determined is called the *solution of the problem.*

9. The unknown number or numbers are found in arithmetic by one or more of the processes of addition, subtraction, multiplication, and division.

10. In the solution of problems in algebra, the equation is employed. In fact, the practical utility of algebra consists in the application of the equation to the solution of problems.

ALGEBRAIC SOLUTION OF PROBLEMS.

1. The sum of two numbers is 84, and the greater number is 6 times the less. What are the numbers?

Let $x =$ the less number;
then $6x =$ the greater number.

Since the *sum* of the numbers is 84,

$$6x + x = 84$$
$$7x = 84$$
$$x = 12, \text{ the less number,}$$
$$6x = 72, \text{ the greater number.}$$

2. A man bought a horse, a carriage, and a harness for \$350. The carriage cost 3 times as much as the harness, and the horse cost twice as much as the carriage. What was the cost of each?

Let $x =$ the number of dollars the harness cost;
then $3x =$ the number of dollars the carriage cost,
and $6x =$ the number of dollars the horse cost.

Since they *all* cost \$350,

$$x + 3x + 6x = \$350$$
$$10x = \$350$$
$$x = \$35, \text{ the cost of the harness,}$$
$$3x = \$105, \text{ the cost of the carriage,}$$
$$6x = \$210, \text{ the cost of the horse.}$$

3. The difference between two numbers is 48, and the greater number is five times the less. Find the numbers.

Let $x =$ the less number;
then $5x =$ the greater number.

Since the *difference* between the numbers is 48,

$$5x - x = 48$$
$$4x = 48$$
$$x = 12, \text{ the less number,}$$
$$5x = 60, \text{ the greater number.}$$

4. A father's age exceeds the age of his son by three times the son's age, and the sum of their ages is seventy-five years. What is the age of each?

Let $x =$ the son's age;
then $3x =$ the difference between their ages,
and $4x =$ the father's age.

Since the *sum* of their ages is 75 years,

$$4x + x = 75$$
$$5x = 75$$
$$x = 15, \text{ the son's age,}$$
$$4x = 60, \text{ the father's age.}$$

TO THE TEACHER.—During the study of the following problems, teach very carefully definitions, principles, and rules, 1 . . . . . . 55. Be sure that students know the law of the algebraic notation, and the signification of coefficients and exponents.

PROBLEMS.

1. A man paid thirty-five dollars for a coat and shoes. If the coat cost 6 times as much as the shoes, what was the cost of the shoes?

2. The sum of two numbers is 126, and the larger number is 8 times the smaller. Find the larger number.

3. A man sold a horse and carriage for $ 500, receiving three times as much for the horse as for the carriage. How much did he receive for the carriage?

4. A has three times as many sheep as B, and both have one hundred eighty. How many sheep has A?

5. There are 795 pupils in a school, and there are twice as many girls as boys. How many girls are there?

6. An estate of twenty-four thousand dollars was divided between a son and daughter, the son having 3 times as much as the daughter. How much did the son receive?

7. The greater of two numbers is six times the smaller, and their difference is 425. Find the larger number.

8. A, B, and C own 750 acres of land. B owns three times as many acres as A, and C owns one-half as many acres as A and B together. How many acres has C?

9. My house cost 3 times as much as the lot. If the house cost five thousand dollars more than the lot, what did the house cost?

10. A man's salary was doubled each year for three years. If he received five thousand two hundred fifty dollars in the three years, what was his salary the second year?

11. A farmer raised 1512 bushels of grain. He had one-fifth as many bushels of rye as oats, and twice as many bushels of corn as oats and rye together. How many bushels of rye and oats did he have?

12. A and B are 81 miles apart. They travel toward each other until they meet, A traveling twice as fast as B. How far does B travel?

13. A and B own a farm worth $ 16,800. A has 3 times as much invested as B. How much has A invested?

14. A is six times as old as B, and the difference between their ages is 65 years. Find the age of B.

15. A man paid four times as much for a farm as he paid for a house. If the farm cost $ 8400 more than the house, how much did he pay for both?

16. In a mixture of 192 bushels of corn and oats, there are twice as many bushels of corn as of oats. How many bushels of corn are there in the mixture?

17. A man earned five times as much as his son. If the son earned six hundred forty-three dollars less than his father, how much did the son earn?

18. A has twice as many sheep as C, and B has four times as many as C. If all have 595, how many has A?

NUMERICAL VALUES.

Find the numerical value of the following expressions when $a=1$, $b=2$, $c=3$, $d=4$, $e=0$, $m=\frac{1}{2}$, and $n=\frac{1}{3}$:

1. $5a+4b+3c+8d.$
2. $8d-7a+8m+6c.$
3. $9c-9n+8d+7c.$
4. $5c+4b-6n+6m.$
5. $7a+5d-3n+6e.$
6. $9d-8e+4m-3a.$
7. $8b+6c-2d+9c^2.$
8. $8m+4e+5d-9a^2.$
9. $7d+6m+8c-9n^2.$
10. $9n+8c-7e+8m.$
11. $3d+4c-2a^2+4m^2.$
12. $4c+6e+8b^3+9a^3.$
13. $2ab+8m-3n^2+2c^2.$
14. $5ad+8a^2-2b-8m^2.$
15. $4cn+2d^3-7e-4m^2.$
16. $6am+9a^2+2bc-3m^2.$
17. $bcd-9n^2-4a^2+6am.$
18. $4ad-6n^2+2dm-2b^2.$
19. $abc+2d^3-6mn+2c^2.$
20. $3cd-3a^2-3mn+4b^2.$
21. $bcd-5e^2-4m^3+3cn.$
22. $5ab+6c^2-4dm+8a^3.$
23. $6b^3-adm+5bc-6n^2.$
24. $7d^3-4cd+5b^3-6am.$
25. $8a^4+6mn-2b^2+bcd.$
26. $8b^3-5am+9bn-2ac.$
27. $5cd-8m^3+9a^3-6cn.$
28. $8ad-7e^2-3bm+2dn.$
29. $4bc+7d^2-9n^3+7ab.$
30. $7a^4+9ad-2bm+5an.$
31. $6bn+5e^3-7cm^2+8ad.$
32. $5bd+6a^3c-4bm+4dn.$
33. $6m+abc+b^2d^3-8d^2m^3.$
34. $emn+9d+8e^3n^2-a^4b^3.$
35. $5b^4m^2-9n+bcm+c^4n^2.$
36. $d^3m^2+5c-4e^2n^3+acd.$
37. $cdm+3c^3n^2+a^4b^5-7e.$
38. $8m+5b^3d^2-a^5c^3-men.$
39. $a^3c^3+7m+4d^2m^3-ben.$
40. $bcd+9c-4a^2m^2+b^3c^3.$
41. $9n+cdm+d^4e^4-2a^3b^2.$
42. $c^3d^2-acn+8m+2d^4m^2.$
43. $3c^3d^2+emn+3n+a^4d^3.$
44. $7d^3m^2+3n-4e^2m^3+ac.$

45. $4c^3d^2 - bm^2n \times a^2e^2 - 4a^3bcd + c^3d^2m - 8c^3 + 2m^4.$

46. $a^4bd \times c^3n^2 + 8b^3c^3 - b^2c^3mn - 3d^2 + 4n^2 - 5a^2ecd.$

47. $ab^2d + 2c^3n^2 \times a^2b^3 - 8d^2 + 4b^3 + a^4b^3c^2d + 9ce^2mn^3.$

48. $cd^2m - 8a^2b^3 \div d^2m^2 + 9c^2 \times 2b^2 + 6a^3dm^2n - b^2c^2d^2n.$

49. $9b^2n + (a^2d^2m^2 + cm)b^2c - emn - (2b^2d^2 - 7c^4n^3 - 8cm)n.$

50. $m^3 \times d^3 + 7bcm + 9a^3c^3 - a^2c^2e^2 + c^2d^2 + b^4m + a^5 \times d^3.$

51. $7bde + 8bc^3m^2 + 6a^3b^2 \times d^2m + b^2c^2 + m^2 - 5c^3dn^2 + 8c^2m^3.$

52. $5c^3d - 9a^4b^3 + 4d^2m^2 + 9c^2n^2 \times 7d^2m^2 - abc \times dem + 4a^4bc^2.$

53. $6abd - (36mn - 4a^5) + 4b(3c^3n^2 + abd) + (4am + c^2m)ad^2.$

54. $7c^2d^2 - n(4abm + b^5) - c^2n^2 \times 7ad - (250 + d^2 \times 8)n - 4a^2n^2.$

55. $5bcd - (2a^2d^2 - b^2m)n + 9d^3m + 4a^2c^2 + m^2n^2 - 5be^2 \times 7c^2d^3.$

56. $3a^2d^2m^2 + e(4ac + bd) + 5b^2dm^2 - a^4d^3m^2 + (m^2 + n^2)c^2d^2.$

Find the numerical value of the following expressions when $a = \frac{1}{2}$, $b = \frac{1}{3}$, $c = \frac{1}{5}$, $d = 2$, $m = 10$, $n = 5$, and $x = 1$:

1. $\dfrac{a+x}{a-c}.$

2. $\dfrac{b+c}{x-c}.$

3. $\dfrac{n-1}{b+x}.$

4. $\dfrac{1}{a} + \dfrac{1}{b}.$

5. $\dfrac{a}{c} - \dfrac{b}{2}.$

6. $\dfrac{n+x}{a-b}.$

7. $\dfrac{x^2+6}{a^2+b}.$

8. $\dfrac{n^2-9}{a^2-c}.$

9. $\dfrac{n}{a} + \dfrac{1}{b^2}.$

10. $\dfrac{x^2+n^2}{a^2+b^2}.$

11. $\dfrac{m^2+n^2}{d^3+b}.$

12. $\dfrac{d^2+x^3}{a^2+c^2}.$

13. $\dfrac{2a+3b}{d^2+x^2}.$

14. $\dfrac{5m+5c}{a^3+b^2}.$

15. $\dfrac{5n^2}{b^2} - \dfrac{6d^2}{ab^2}.$

16. $\dfrac{8a^2}{c^2} - \dfrac{6n^2}{d^2}.$

17. $\dfrac{c^2m^2}{4b^2} + \dfrac{3x^3}{a^3}.$

18. $\dfrac{10c+10d}{3b^2+5c^2}.$

19. $\dfrac{2m^2}{a^2d^2} - \dfrac{4nx}{3b^3}.$

20. $\dfrac{cdm}{ab} + \dfrac{4mx}{bc}.$

21. $\dfrac{5cn}{b^2c^2} - \dfrac{3dm}{2a^3}.$

Find the numerical value of the following expressions when $a = 10$, $b = 8$, $c = 5$, $d = 4$, $e = 3$, and $x = 1$:

1. $a(b + c)$.	2. $(e - x)ac$.	3. $abx(a + c)$.
4. $(a - e)d^2$.	5. $dx^2(c + e)$.	6. $(a - c)5ex$.
7. $(b - c)ax^2$.	8. $(e + x)cd^2$.	9. $(a + 2d)cx^2$.
10. $a^2e^2(a - b)$.	11. $(a - x)d^2e^2$.	12. $2c^2x^2(c + x)$.
13. $(a + c)(b - d)^2$.	14. $e^2x^2(a - b)^2$.	15. $2dex(c - d)^2$.
16. $4x(b - c)(e + d)$.	17. $(x^2 + c^2)2ce^2$.	18. $(a^2 - b^2)3ce^2x^2$.

Find the numerical value of the following expressions when $a = 8$, $b = 6$, $c = 5$, $d = 3$, $e = 0$, and $m = \frac{1}{4}$:

1. $(a + b)c - d$.	2. $(a + b)(c - d)$.
3. $(b + c)2d^2 + 4a^2m$.	4. $(c^2 + d^2)(b - d)$.
5. $(a^2 + c^2)(m^2 - de^2)$.	6. $3e(a + m)(b - d)$.
7. $(a - b + c)(d - m)$.	8. $(b + m)d(b - c)4a^2$.
9. $(b^2 + c^2 + d^2) - (a - m)$.	10. $(c^2 + d^2)(b^2 - c^2)2am$.
11. $(ac + b + d) - m^2(c^2 - d^2)$.	12. $3bdm(a - c + d) - 5a^2e^2$.

Find the numerical value of the following expressions when $a = 8$, $b = 7$, $c = 6$, $d = 5$, $e = 4$, $m = 3$, $n = 2$, and $x = 1$:

1. $(a^2 + n^2)(c - m)^2$.	2. $(b + x)m(c^2 - m^2)$.
3. $(em + x^2)n(b - d)^2$.	4. $(d^2 - n^2)(b - 5x^2)^2$.
5. $(m^2 - n^2)(9c - e^2)^2$.	6. $(b^2 - x^2)8(9d - c^2)$.
7. $a^n + b^{m-1} + d^{m-n}e^2$.	8. $d^m - a^{b-d} + c^{n+1}ex^2$.
9. $x^m + d^{n+1} + e^{m+1}x^{2n}$.	10. $a^2 - 2(m - x)3(b - d)$.
11. $9a - (a - c)5(e^2 - m^2)$.	12. $a^{n+1} - m(a - b)c(d - 2x)$.

TO THE TEACHER. — Teach 56 . . . . 62.

ADDITION.

1.	2.	3.	4.	5.
$5ax$	$-7xy$	$6ac$	$3ab$	$5bc$
$4ax$	$-2xy$	$-2ac$	$-9ab$	$-bc$

6.	7.	8.	9.	10.
$4ab$	$-7cd$	$-2ax$	$5ac$	$9an$
ab	$-cd$	$-8ax$	ac	an
$3ab$	$-5cd$	$-ax$	$2ac$	$7an$
ab	$-2cd$	$-5ax$	$9ac$	an

11.	12.	13.	14.	15.
$4cx$	$-am$	$6bc$	cd	$5xy$
$3cx$	$4am$	$-2bc$	$-9cd$	$9xy$
cx	$-9am$	$5bc$	$5cd$	$-xy$
$5cx$	$-3am$	bc	$-8cd$	$8xy$

16.	17.	18.	19.	20.
$7ad$	$-9xy$	$8ab$	ac	$9bc$
ad	$4xy$	$-4ab$	$-7ac$	$-4bc$
$4ad$	$-xy$	$-ab$	$-ac$	bc
ad	$2xy$	$7ab$	$2ac$	$-2bc$
$5ad$	$-6xy$	$-4ab$	$3ac$	bc
$2ad$	xy	ab	$-9ac$	$-4bc$
$6ad$	$5xy$	$-2ab$	$4ac$	$-2bc$

21.	22.	23.	24.	25.
$2ab$	$3mn$	abc	$5ax$	abd
$3cd$	$-2ab$	xyz	$-3cy$	$-4xy$

26.	27.	28.	29.	30.
$5ax$	$7ac$	$15x$	acx	$4ab$
$2cd$	$-bd$	$-2ay$	$22y$	$-3cd$

31.	**32.**	**33.**	**34.**	**35.**
$7ax^2$	$5mn^2$	$-7cd$	$3cx$	$-7x^2y$
$-9ax^2$	$-mn^2$	$4cd$	$-2cx$	$9x^2y$
$-6ax^2$	$-8mn^2$	$-cd$	$9cx$	$5x^2y$
$8ax^2$	$4mn^2$	$-8cd$	$-8cx$	$-8x^2y$
$-7ax^2$	$6mn^2$	cd	$6cx$	$-4x^2y$
ax^2	$3mn^2$	$-6cd$	$-3cx$	$9x^2y$
$5ax^2$	$-mn^2$	$2cd$	$-4cx$	$-5x^2y$

36.	**37.**	**38.**	**39.**	**40.**
$7bc^2$	$8ax^2$	$-4ax$	xy	$-4a^2d$
$-3bc^2$	$-3ax^2$	$-ax$	$-6xy$	a^2d
$6bc^2$	$5ax^2$	$3ax$	$5xy$	$-2a^2d$
bc^2	$-2ax^2$	$-7ax$	$-xy$	$-a^2d$
$9bc^2$	$7ax^2$	$5ax$	$4xy$	$9a^2d$
$-5bc^2$	$-ax^2$	$-9ax$	$-2xy$	$-7a^2d$
$-bc^2$	ax^2	ax	$-9xy$	$3a^2d$

41.	**42.**	**43.**	**44.**	**45.**
acx	$-5ab$	$3mx$	$-2ax$	$-6ac$
$-2bd$	$-3cd$	$-ny$	$-5by$	$-by$

46.	**47.**	**48.**	**49.**	**50.**
$2ab$	$-2bd$	$-4ax$	$7ab$	$-cd$
$-xy$	$3ac$	$-by$	-24	$3ab$

51.	**52.**	**53.**	**54.**	**55.**
$5ax$	$-3cd$	$5bx$	$-5ab$	$-7xy$
$-ax$	$-2ab$	$-xy$	$-2cd$	$-2bc$
$2by$	$7cd$	$-4bc$	$8ab$	$6xy$

56.	**57.**	**58.**	**59.**	**60.**
$5ac$	$4ab$	$-2ax$	$4cd$	$-cy$
$-ax$	$-3ab$	$-ac$	$-8cd$	$5xy$
$4ac$	-12	$3ax$	ab	$-4xy$

TO THE TEACHER. — Teach how to add or unite terms that are partly similar. See 63.

Add the following with reference to the similar parts:

1.	2.	3.	4.	5.
ax bx	ay cy	ax x	mx $-nx$	mx $-x$

6.	7.	8.	9.	10.
$-y$ ay	$2x$ $2y$	bx $-x$	a ax	ac $-bc$

11.	12.	13.	14.	15.
ax x ax	ac $4c$ $-9c$	$4y$ cy $-y$	ax bx $-x$	$2ad$ $-3bd$ $3ad$

16.	17.	18.	19.	20.
ax^2 $-3bx^2$ $-2ax^2$ $5ax^2$ 5 + ay	$6ay$ $3by$ $-2ay$ $-9by$	$2axy$ $7xy$ $-axy$ $-8xy$	$4ay^2$ $-2by^2$ $-ay^2$ $-2ay^2$	$-3axy^2$ $7bxy^2$ $4axy^2$ $-8bxy^2$

21.	22.	23.	24.	25.
$a(c+d)$ $b(c+d)$	$a(x-y)$ $(x-y)$	$a(x+y)$ $-c(x+y)$	$b(x-1)$ $-(x-1)$	$3a(x+y)$ $(x+y)$

26.	27.	28.	29.	30.
$a(b+4)$ $3(b+4)$	$a(c-1)$ $-(c-1)$	$c(x+y)$ $-3(x+y)$	$a(b-2)$ $(b-2)$	$(x-1)$ $-c(x-1)$

31.	32.	33.	34.	35.
$2a(x-1)$ $4(x-1)$ $-a(x-1)$ $-(x-1)$	$-(b+c)$ $5a(b+c)$ $5(b+c)$ $-2a(b+c)$	$3a(x-3)$ $a(x-3)$ $-2a(x-3)$ $(x-3)$	$-b(c+2)$ $3b(c+2)$ $-(c+2)$ $2b(c+2)$	$4a(x+y)$ $6(x+y)$ $-2a(x+y)$ $-(x+y)$

To the Teacher. — Teach addition of polynomials.

31.	32.	33.	34.	35.
$7ax^2$	$5mn^2$	$-7cd$	$3cx$	$-7x^2y$
$-9ax^2$	$-mn^2$	$4cd$	$-2cx$	$9x^2y$
$-6ax^2$	$-8mn^2$	$-cd$	$9cx$	$5x^2y$
$8ax^2$	$4mn^2$	$-8cd$	$-8cx$	$-8x^2y$
$-7ax^2$	$6mn^2$	cd	$6cx$	$-4x^2y$
ax^2	$3mn^2$	$-6cd$	$-3cx$	$9x^2y$
$5ax^2$	$-mn^2$	$2cd$	$-4cx$	$-5x^2y$

36.	37.	38.	39.	40.
$7bc^2$	$8ax^2$	$-4ax$	xy	$-4a^2d$
$-3bc^2$	$-3ax^2$	$-ax$	$-6xy$	a^2d
$6bc^2$	$5ax^2$	$3ax$	$5xy$	$-2a^2d$
bc^2	$-2ax^2$	$-7ax$	$-xy$	$-a^2d$
$9bc^2$	$7ax^2$	$5ax$	$4xy$	$9a^2d$
$-5bc^2$	$-ax^2$	$-9ax$	$-2xy$	$-7a^2d$
$-bc^2$	ax^2	ax	$-9xy$	$3a^2d$

41.	42.	43.	44.	45.
acx	$-5ab$	$3mx$	$-2ax$	$-6ac$
$-2bd$	$-3cd$	$-ny$	$-5by$	$-by$

46.	47.	48.	49.	50.
$2ab$	$-2bd$	$-4ax$	$7ab$	$-cd$
$-xy$	$3ac$	$-by$	-24	$3ab$

51.	52.	53.	54.	55.
$5ax$	$-3cd$	$5bx$	$-5ab$	$-7xy$
$-ax$	$-2ab$	$-xy$	$-2cd$	$-2bc$
$2by$	$7cd$	$-4bc$	$8ab$	$6xy$

56.	57.	58.	59.	60.
$5ac$	$4ab$	$-2ax$	$4cd$	$-cy$
$-ax$	$-3ab$	$-ac$	$-8cd$	$5xy$
$4ac$	-12	$3ax$	ab	$-4xy$

TO THE TEACHER. — Teach how to add or unite terms that are partly similar. See 63.

Add the following with reference to the similar parts:

1.	**2.**	**3.**	**4.**	**5.**
ax	ay	ax	mx	mx
bx	cy	x	$-nx$	$-\ x$
6.	**7.**	**8.**	**9.**	**10.**
$-y$	$2x$	bx	a	ac
ay	$2y$	$-\ x$	ax	$-bc$
11.	**12.**	**13.**	**14.**	**15.**
ax	ac	$4y$	ax	$2ad$
x	$4c$	cy	bx	$-3bd$
ax	$-9c$	$-\ y$	$-\ x$	$3ad$
16.	**17.**	**18.**	**19.**	**20.**
$\cancel{ax^2}$	$6ay$	$2axy$	$4ay^2$	$-3axy^2$
$-3bx^2$	$3by$	$7xy$	$-2by^2$	$7bxy^2$
$-2ax^2$	$-2ay$	$-\ axy$	$-\ ay^2$	$4axy^2$
$5\cancel{ax^2}$	$-9by$	$-8xy$	$-2ay^2$	$-8bxy^2$
5 + ax				
21.	**22.**	**23.**	**24.**	**25.**
$a(c+d)$	$a(x-y)$	$a(x+y)$	$b(x-1)$	$3a(x+y)$
$b(c+d)$	$(x-y)$	$-c(x+y)$	$-\ (x-1)$	$(x+y)$
26.	**27.**	**28.**	**29.**	**30.**
$a(b+4)$	$a(c-1)$	$c(x+y)$	$a(b-2)$	$(x-1)$
$3(b+4)$	$-\ (c-1)$	$-3(x+y)$	$(b-2)$	$-c(x-1)$
31.	**32.**	**33.**	**34.**	**35.**
$2a(x-1)$	$-\ (b+c)$	$3a(x-3)$	$-\ b(c+2)$	$4a(x+y)$
$4(x-1)$	$5a(b+c)$	$a(x-3)$	$3b(c+2)$	$6(x+y)$
$-a(x-1)$	$5(b+c)$	$-2a(x-3)$	$-\ (c+2)$	$-2a(x+y)$
$-\ (x-1)$	$-2a(b+c)$	$(x-3)$	$2b(c+2)$	$-\ (x+y)$

TO THE TEACHER. — Teach addition of polynomials.

Express the following in their simplest form:

1. $2x-4y-6x-5z+3y+3z-2y+5x-z+3y.$
2. $4b+3c-2d-5c-2b+8d+3b+3c-c-6d.$
3. $5x-3y-4z+5y-3z-3u-2x+9z-6y+4x.$
4. $3a-4c+5d+7c-4e-2a-2d-5f-3c+3e.$
5. $5b-3c+6d-4e-9d+5c+3d-2b+3f-6c.$
6. $2x+3y+2y-3z-3x+2z+2x+3u-4y-6z.$
7. $6a+4b-7c-5b+3d-2a+7c-3d+7b+5m.$
8. $7xy-3y+z+y+xy-z-6xy+z-3x-y+7z-y.$
9. $5ac+2d-e-d+ac-e-3ac-2f+e+d+5e-d.$
10. $5bx-3cx-4dx+xy+cx+5dx+2cx-2xy-4bx.$
11. $7a-5b+4c-2c-a-d+5c+5b-2a+6d-3a.$
12. $5ab+4ab^2-5c-9ab^2+3c-4ab+4c+3ab^2-6c.$
13. $4ac-2ab-8ay+4ab-ax-4ay-4ac+7ay-ab.$
14. $3a+5b-4c+6c-a+3d+2a-c-6b-2d-3a.$
15. $7x^3y^2-3x^2y^3-7z^2+5x^2y^3-9x^3y^2-2u^2+2x^3y^2+9z^2.$
16. $7ac-8ax+5ay+3ax-3xy-4ac-4ay+5ax-ac.$
17. $9a^2b^3+4a^3b^2-3b^2-3a^3b^2-2c^2-7a^2b^3+3b^2-2a^2b^3.$
18. $5ab-4ac+7ax-4xy-3ab-4ax+4ac-2ab+xy.$
19. $7ab-5ac+3ad+2ac-2ae-5ab+2ac-4ad-ab.$
20. $5ac-7ax+4ax-3ay-3ac-2ay+7ax+6ay+ac.$
21. $5ax+2ay-7az-3xy+4ay+3az-7ay+5az+ax.$
22. $3ab+5ac-3ac-5ad+4ad+7ab-8ab-3ac-ad.$
23. $4ac-7ae+7ad-3mn-7ac-5ad+9ae+3ac+ad.$
24. $5xy-3xy^2+3z-8xy^2-5z-4xy+8z-6xy^2-7z+3xy.$
25. $4ab-5ab^2-5c-2ab+2c+7ab^2-3c-3ab^2+7c+5ab.$

1. Add $3x^3 - 5x^2 + 2x - 6$, $x^2 + 4 - 6x + 2x^3$, $3x^2 - 3x - x^3 + 8$, $4x + 4x^3 - 4 - 2x^2$, and $3x^2 + 5x - 2 - 7x^3$.

2. Add $4a^3 - 8a^2 + 3a - 5$, $5a^2 + 9 - 7a - a^3$, $-a^2 + 7a^3 - 7 + 4a$, $6a^2 - 3a - 8a^3 + 6$, and $a - 4a^2 - 3 + 3a^3$.

3. Add $ab - \frac{1}{2}ac + \frac{1}{4}ad - \frac{1}{3}ae$, $ac + \frac{5}{6}ae - \frac{1}{2}ad + \frac{3}{4}ab$, $ad + 2ac + \frac{1}{2}ae + \frac{1}{4}ab$, and $ae + ac + ad + ab$.

4. Add $2(a+b)+7c$, $(a+b)-2c$, $c-3(a+b)$, $(a+b)-5c$, $-c-(a+b)$, and $8c+2(a+b)$.

5. Add $\frac{1}{4}ax - \frac{1}{2}ay + \frac{2}{3}bx - 2by$, $ay + by + \frac{1}{2}ax$, $ay - \frac{1}{2}by + ax - bx$, and $\frac{2}{3}by - \frac{3}{4}ay + \frac{3}{4}bx + \frac{1}{3}ax$.

6. Add $x^3 + 3x^2 - 5x + 7$, $-x^2 - 4 + 2x - 3x^3$, $-x + 5x^3 + 3 - 5x^2$, $4x^2 - 1 - 4x - x^3$, and $8x + 2 - 2x^3 - x^2$.

7. Add $2(a+c)+3(b-c)-bc$, $4bc-(a+c)-(b-c)$, $7(a+c)-2bc+5(b-c)$, and $8(a+c)-bc-(b-c)$.

8. Add $\frac{1}{3}a^2x - \frac{1}{2}ax^2 + \frac{1}{4}ax - \frac{2}{3}ay$, $a^2x + 2ay - ax$, $\frac{1}{2}ax - \frac{1}{2}a^2x + ax^2$, and $2a^2x - \frac{1}{3}ax^2 - \frac{5}{6}ay$.

9. Add $a(a+b) + a(a-b) - a(a+c)$, $-a(a-b) + (a+b) + a(a-c) + a(a+c)$, and $b(a-b) + 2(a+c)$.

10. Add $a(a+b) + 3(b+c) + (b-c)$, $(b+c) + 3(a+b) + b(b-c)$, and $a(b+c) - (a+b) - 5(b-c)$.

11. Add $\frac{7}{8}x^3y^2 + x^2y^3 - \frac{3}{8}xy + \frac{1}{2}xz$, $xy - x^3y^2 - xz$, $\frac{3}{4}xz - \frac{3}{4}xy - \frac{2}{3}x^2y^3$, and $xy + \frac{5}{6}x^2y^3 + \frac{3}{4}x^3y^2$.

12. Add $ab + ac + ae - 2bc$, $-ac + bc - ab$, $\frac{5}{6}ac - \frac{1}{3}bc + ab - \frac{5}{4}ae$, and $ae - 2ac + \frac{1}{2}bc - \frac{2}{3}ab$.

13. Add $(a+c) + 2a(b+c)$, $b(b-c) + a(a+c) - (b+c)$, $(a+c) - (b-c) - a(b+c)$, and $4(b-c) + (a+c)$.

EXERCISES IN ALGEBRAIC EXPRESSION.

1. Write six times the product of x and y, increased by three times the square of x.

2. Write three times the sum of a square and b square, diminished by five times the product of a, b, and c.

3. If a represents an odd number, what will represent the next larger odd number?

4. If x represents an even number, what will represent the next smaller even number?

5. If a man can perform a piece of work in x days, what part of it can he do in one day?

6. Write an expression for the sum of three consecutive numbers of which x is the smallest.

7. How many square yards are there in a ceiling which is x feet long and y feet wide?

8. What is the interest on five hundred dollars for a years at x per cent per annum?

9. Write an expression for the sum of four consecutive even numbers of which x is the largest.

10. If a man can perform a piece of work in x days, what part of it can he do in four days?

11. When a represents an integer, does $2a - 1$ represent an even number or an odd number? Show why.

12. At six per cent per annum, what is the interest on three thousand dollars for x years?

13. Write an expression for the sum of five consecutive odd numbers of which x is the middle one.

14. In eight years a man will be x years old. How old was he eight years ago?

15. A man bought x sheep at a dollars a head and had b dollars left. How much money had he at first?

EQUATIONS AND PROBLEMS.

To the Teacher.—Teach how to transpose a term from one member of an equation to the other, also how to solve such equations as the following:

1. $4x-4=2x+8.$	2. $5x+3=2x+9.$
3. $9x-8=4x+7.$	4. $7x+4=3x+8.$
5. $6x-3=5x+4.$	6. $8x+1=6x+9.$
7. $7x-6=4x+9.$	8. $3-3x=9-5x.$
9. $1+2x=9-2x.$	10. $5-4x=7-6x.$
11. $3x-7=8-2x.$	12. $1-7x=9-9x.$

1. My house and lot cost $17,500, the house costing 4 times as much as the lot. Find the cost of the house.

2. Twice a number increased by 68 is equal to three times the number diminished by 57. Find the number.

3. The sum of two numbers is 112, and their difference is 36. What is the larger number?

4. A horse and carriage cost $365. If the horse cost $85 more than the carriage, what was the cost of the horse?

5. A, B, C, and D have 280 sheep. B has twenty more than A, C has twenty more than B, and D has twenty more than C. How many have A and B?

6. A man paid $\frac{1}{4}$ of a debt and still owes $90 more than he paid. How much of the debt did he pay?

7. The sum of the ages of father, mother, and son is 105 years. The mother is twice as old as the son, and five years younger than the father. Find the father's age.

8. A has twice as many acres of land as B, and B has three times as many acres as C. If together they have 2600 acres, how many acres have B and C?

9. Separate 148 into two parts such that the greater shall exceed the less by 34.

10. The sum of two numbers is 259, and their difference is five times the smaller. Find the larger number.

11. A man lost $\frac{1}{6}$ of his money, but he still has \$5800 more than he lost. How much has he left?

12. Three farmers sold 2080 bushels of potatoes. A sold $\frac{1}{4}$ as many bushels as B, and C sold $\frac{3}{5}$ as many bushels as A and B together. How many bushels did B sell?

13. A father and son earn \$107 a month. If the son's wages were doubled, he would receive only \$11 less than his father. How much does the son receive?

14. A man built a house costing five times as much as the lot. If the lot cost fourteen thousand dollars less than the house, how much did the lot cost?

15. Three men raised 1925 bushels of corn. A had three times as many bushels as B, and 175 bushels more than C. How many bushels did A and B raise?

16. A has one-sixth as much money as B, and C has $2\frac{1}{7}$ times as much as A and B. If C's money exceeds B's by \$4500, how much has B?

17. The sum of the ages of father and son is 72 years, and the difference between their ages is twice the son's age. Find the age of the son.

18. A man has four thousand fifty dollars in four banks. He has twice as much in the first bank as in the second, twice as much in the third as in the first and second, and twice as much in the fourth as in the other three. How much money has he in the third bank?

To the Teacher.—Teach 65 . . . . 71.

SUBTRACTION.

1.	**2.**	**3.**	**4.**	**5.**	**6.**
$7a$	$-9b$	$3c$	$-6x$	$9a$	$-7b$
$3a$	$-2b$	$-5c$	$-7x$	$+4a$	$-8b$
7.	**8.**	**9.**	**10.**	**11.**	**12.**
$4x$	$-3c$	$9a$	$-7b$	$-8c$	$5b$
$-5x$	$+8c$	$-8a$	$+6b$	$-3c$	$+8b$
13.	**14.**	**15.**	**16.**	**17.**	**18.**
$9c$	$-7x$	a	$-5c$	$7b$	$-6x$
$-c$	$+x$	$-9a$	$+6c$	$+2b$	$-8x$
19.	**20.**	**21.**	**22.**	**23.**	**24.**
a	$-c$	$8x$	$-4a$	$-7d$	$4e$
$-2a$	$+7c$	$-7x$	$+3a$	$-3d$	$+9e$
25.	**26.**	**27.**	**28.**	**29.**	**30.**
$8x$	$-7c$	$4b$	$-4a$	$8x$	$-5d$
$-3x$	$+2c$	$-8b$	$-5a$	$-4x$	$7d$
31.	**32.**	**33.**	**34.**	**35.**	**36.**
$6c$	$-2a$	$7y$	$-9d$	$-7e$	$3a$
$7c$	$-6a$	$6y$	$-8d$	$5e$	$-8a$
37.	**38.**	**39.**	**40.**	**41.**	**42.**
$7b$	$-2x$	d	$-3c$	$9a$	$-4e$
b	$-x$	$7d$	$-4c$	$-5a$	$9e$
43.	**44.**	**45.**	**46.**	**47.**	**48.**
d	$-a$	$6x$	$-8b$	$-9e$	$5c$
$2d$	$-5a$	$5x$	$-7b$	$7e$	$-9c$

49. $7ac$ $2ac$	50. $-7x^2$ $-3x^3$	51. $6a$ $5a$	52. $-3bc$ $-4bc$	53. $9b^3$ $-4b^3$	54. $-4ab^2$ $7ab^2$
55. $3bc$ $8bc$	56. $-3c^2$ $-8c^2$	57. $3x$ $4x$	58. $-7ab$ $-6ab$	59. $-8x^3$ $3x^2$	60. $4bc^2$ $-7bc^2$
61. $9ab$ ab	62. $-6a^2$ $-a^2$	63. $2b$ b	64. $-ac$ $-2ac$	65. $7a^3$ $-a^3$	66. $-ac^2$ $8ac^2$
67. ac $9ac$	68. $-a$ $-7a^2$	69. a $2a$	70. $-2ax^2$ $-ax^3$	71. $-7b$ b	72. xy^3 $-8xy^2$
73. $8bc^2$ $3bc^2$	74. $-9b^2$ $-2b^3$	75. $9c$ $2c$	76. $-5bc^3$ $-6bc$	77. $8c$ $-7c^2$	78. $-7ax^2$ $9ax^2$
79. $4ab^3$ $9ab^3$	80. $-7x$ $-9x^3$	81. $7b$ $8b$	82. $-8ac^2$ $-7ac^2$	83. $-8a$ $6a$	84. $8ab^2$ $-9ab^2$
85. $6xy^2$ xy^2	86. $-8c$ $-c^3$	87. $2x$ x	88. $-ax^3$ $-2ax^3$	89. $6x$ $-x^2$	90. $-xy^3$ $7xy^3$
91. ax^3 $7ax^2$	92. y $-6y^3$	93. a $4a$	94. $-2xy^3$ $-xy^3$	95. $-9c$ c^2	96. $7ac^2$ $-8ac^2$

TO THE TEACHER.—Teach 72.

Subtract with reference to the similar parts:

1.	**2.**	**3.**	**4.**	**5.**	**6.**
ax	$-ay$	ac	$-ax$	ac	y
bx	$-2y$	c	$-x$	$-bc$	$-cy$
7.	**8.**	**9.**	**10.**	**11.**	**12.**
bc	$-ax$	d	bx	$-c$	ay
$3c$	$-bx$	cd	$-x$	$-bc$	$-2y$
13.	**14.**	**15.**	**16.**	**17.**	**18.**
ac	$-ax$	by	ac	$-am$	c
bc	$-3x$	y	$-bc$	$-m$	$-ac$
19.	**20.**	**21.**	**22.**	**23.**	**24.**
ax	$-ac$	y	cx	$-n$	ad
$4x$	$-bc$	cy	$-x$	$-cn$	$-4d$
25.	**26.**	**27.**	**28.**	**29.**	**30.**
am	$-bc$	ad	bx	$-ay$	n
cm	$-5c$	d	$-dx$	$-y$	$-an$
31.	**32.**	**33.**	**34.**	**35.**	**36.**
ad	$-cm$	n	ac	$-x$	bc
$5d$	$-am$	mn	$-c$	$-cx$	$-5c$
37.	**38.**	**39.**	**40.**	**41.**	**42.**
cy	$-cx$	ay	mx	$-ad$	b
dy	$-4x$	y	$-nx$	$-d$	$-ab$
43.	**44.**	**45.**	**46.**	**47.**	**48.**
be	$-dy$	c	am	$-x$	ay
$6e$	$-cy$	ac	$-m$	$-bx$	$-7y$

1. From $a-b+c+4$ subtract $b+c-a$.
2. Subtract $u+y+x-z$ from $x+2y-z$.
3. From $a+c+2x-y$ subtract $2x-5+a$.
4. Subtract $2c+y-2x$ from $3c+d+x+6$.
5. From $5b+2c+d$ subtract $c+8-4d-6b$.
6. Subtract $5z-4+6x$ from $7x-8y+5z-5$.
7. From $7c-8d-5e$ subtract $3d+f-4c-4e$.
8. Subtract $d-4+3ab-6c$ from $8ab-5c+4d$.
9. From $4ax+6ay-z+8$ subtract $5ay-z+9ax$.
10. Subtract $8de+7+5bc$ from $6bc-4b+8de+f$.
11. From $4cx+7by-2xy-9$ subtract $7by+8+3cx$.
12. Subtract $3ab+6-7ac-3ax$ from $4ab-6ac-am$.
13. From $5ax-7ay-3xy-23z$ subtract $4xy-7-7ay$.
14. Subtract $5e-8be-2bc$ from $4bc+2bd-7be-8de$.
15. From $6am-5an+4ar-7rs$ subtract $125+6am-9an$.
16. Subtract $4ax-2xy+7ab$ from $4ax-5ac-3xy+8ab$.
17. From $4ay-7xy-3by+5xz$ subtract $6xy+6xz-3by$.
18. Subtract $3a^2x-8-2a^3y$ from $3a^2x+3ax-2a^3y-4az$.
19. From $8x^2y-8xy^2+3xz$ subtract $7x^2y-2xz+7-9xy^2$.
20. Subtract $8x^2y-6xy-36x^3$ from $25x^3+9x^2y+7xy-8$.
21. From $7ab-5ab^2+24b^3$ subtract $7ab-6ab^2-7-37b^3$.
22. Subtract $7bc^2-abd+9b^2c$ from $4b^2c+8bc^2-abd-9$.
23. From $8a^2x-8a^2y+187$ subtract $z+7a^2y+243+7a^2x$.
24. Subtract $2xyz-93y^2-8+29x^3$ from $48x^3-7y^2+xyz$.
25. From $16a^2c+ac^2+d$ subtract $30ac^2+7-25d+73a^2c$.
26. Subtract $9xy^2+z-434-22x^3y$ from x^3y+8xy^3-256.

In such examples as the following, write all the polynomials, similar terms in a column, writing those to be subtracted with their signs changed, and then add.

1. From the sum of $2a + b - c + d$ and $3d + 4a - 3b$ subtract $2b + 4 + 4d + 7a$.

2. Subtract $3z - 2u + 2x$ from the sum of $3x - 4y + 3z - u$ and $8 + 5y - x - z$.

3. From $b - c + d - 4e$ subtract the sum of $2d - 7e + 7b - 3c$ and $3e - 6b - d$.

4. Subtract the sum of $2b + 3y - 7x - 5a$ and $5x - b - 4y - a$ from $a + b - 3x$.

5. From the sum of $4x + y - 2z$ and $4u + 3y - 7x - 2z$ subtract $4u + 4y - 5z + 5 - 3x$.

6. Subtract $3a - 2d - b - 4c$ from the sum of $4a - 2b + 3c + d$ and $b - 3d + 5a - 2c$.

7. From $3x - 4y + 3z - u$ subtract the sum of $z + x - 8 - 5y$ and $3z + 2x - 2u$.

8. Subtract the sum of $5m - a - 9n$ and $5b + 5n + a - 4m$ from $a + 4b + 6m - 4n$.

9. From the sum of $4c + d - 3e$ and $7e - f - 9d - 5c$ subtract $e - 2d - 2f - c$.

10. Subtract the sum of $\frac{1}{4}x - \frac{1}{2}y - c$ and $z - \frac{1}{2}c$ from the sum of $\frac{1}{2}a - \frac{1}{3}c - \frac{1}{4}x$ and $\frac{3}{4}y - \frac{1}{2}x + z - \frac{5}{6}a$.

11. From the sum of $m + \frac{1}{4}n - \frac{3}{8}r$ and $r - 4s - n - 2m$ subtract the sum of $8 - 3s - \frac{1}{2}n$ and $\frac{3}{4}r - \frac{1}{4}n$.

12. Subtract the sum of $3a + e + c$ and $\frac{5}{6}d + \frac{3}{4}b$ from the sum of $4a - \frac{7}{8}b - c$ and $2c + \frac{2}{3}d + 2b + e$.

C

13. From the sum of $2ab - ac + 2bc$ and $2ac - bc - 3ab$ subtract the sum of $3ac - 4bc - ab$ and $2abc - 2ab - 2ac$.

14. Subtract the sum of $a^2c - ac^2 + a^2b$, $a^2b - ac^2 - a^2c$, and $ab^2 + a^2c + ac^2$ from $a^2b + a^2c - ac^2$.

15. From the sum of $3x^2 - 2x + 5$ and $4x + 2y - 4$ subtract the sum of $3x - 2x^2 + 3$ and $y - 2x - 2$.

16. From the sum of $3xy - 2xz + yz$ and $xz + 5z^2 - xy$ subtract the sum of $4xz - 2yz$ and $4z^2 + xy + yz - 5xz$.

17. Subtract the sum of $xy^2 - xz^2 - x^2z$ and $xz^2 - x^2y - x^2z$ from the sum of $xy^2 - x^2y - x^2z$ and $x^2y - xz^2 - x^2z$.

18. From $4ab - 3ac + 2bc$ subtract the sum of $3bc + bd - ac$, $3ab - 3bd - bc$, and $bd - 2ac - ab$.

19. From the sum of $2x^2y - 3x^2z + xz^2$, $2x^2z - 2xz^2 - x^2y$, and $2xz^2 - xy^2 - x^2z$ subtract $xz^2 - xy - 2x^2z$.

20. Subtract the sum of $2x - x^2 - y$, $x^3 + 5 - 2x^2$, and $x - x^3 - 4 - y$ from $2x^3 - 4x^2 + 3x$.

21. From the sum of $-a^2b + a^2c - ab^2$ and $ac^2 - 2a^2c + 3a^2b$ subtract the sum of $-a^2c - 2ab^2$ and $2a^2b - 2ac^2 - ab^2$.

22. Subtract $x^2z - xy - 2xy^2$ from the sum of $x^2y - 2xy + xy^2$, $x^2z - 3x^2y - 5xy^2$, and $3xy + x^2z + 2x^2y$.

23. Subtract the sum of $2b^2 - d^2 + 2c^2$, $a^2 + 6 - c^2$, and $2d^2 - b^2 - 5$ from $2a^2 + 3b^2 + c^2 - d^2$.

24. Subtract the sum of $a^2b - 4a^2c + 5ab^2$ and $2a^2b - 2ac^2 - 2a^2c$ from the sum of $2ab^2 + 3a^2b - 5a^2c$ and $4ab^2 - 3ac^2 - a^2b$.

25. Subtract $2xz^2 - 2xy^2 - x^2y$ from the sum of $xy^2 - xz^2 + x^2y - x^2z$, $-3x^2y - 2xy^2$, and $4x^2z + 3xz^2 + x^2y$.

To the Teacher. — Teach 73.

Remove the signs of grouping and simplify:

1. $a - b - (b - 2a).$ **2.** $x - (y - x + 2y).$

3. $a + (b - \overline{2a - b}).$ **4.** $5x + y - (x + y).$

5. $c + d - (-c + d).$ **6.** $a + (-a - \overline{a + b}).$

7. $x - (-x - \overline{x - y}).$ **8.** $a - b - (-a + b).$

9. $3x - y - x - (-y).$ **10.** $c + (-c - \overline{4c - d}).$

11. $a + 4b - (-a + b).$ **12.** $a - 3b - (-a + b).$

13. $7c - (-2c - \overline{c + d}).$ **14.** $7b - (-c - \overline{4b + c}).$

15. $5x - y - (-2x + y).$ **16.** $5x + y - (-4x - y).$

17. $8a - (-4a + \overline{b + c}).$ **18.** $9a - (-\overline{b - 8c + d}).$

19. $5c - (-b + 4c) - 2c.$ **20.** $3m - n - (-n + m).$

21. $8x - \{y - \overline{3z - (y - 5z + x - 6y) - 4z}\}.$

22. $4a - [b - (3c - \overline{a + 2c - 4b}) + a + 3b].$

23. $(3c - \overline{5d + 4 - 7e}) - (2c + \overline{4d + 5} + 3e).$

24. $5a - [3b + \overline{2a - (5 - 5b)} - \overline{4a - 4} - 9b].$

25. $5b - \overline{5c - [5 + 3b - (7 - c + 7b) - 4] - 4c}.$

26. $2x - \{y - \overline{3x + (y - 6z + x)} - \overline{z + 3y - 4z}\}.$

27. $5a - \{7b + \overline{4c - (3a + 8b + c) + 7a}\} - \overline{b - c}.$

28. $7x - \overline{3y + z - [4x + (2z + 4y - 2) - 9x] + y}.$

29. $9c - \{-\overline{4c - (7 - 2a + e)} - \overline{3c + (4a - 2e)} + 6\}.$

30. $x - \{-\overline{2y - (4z + 8x)} - y - \overline{4x + (z - 3y)} - 3x\}.$

31. $9a - [-\{b - \overline{c - (6b - c + 3a - 9c) - 7b + 5a}\}].$

32. $5x - [7x - \{2x - (3x - \overline{4x - 8})\} - (5x - \overline{4x - 7})].$

33. $9c - [-\overline{8d - (5 - 2e + 6d) + 4c}] - \overline{7 - (3e + 5c)}.$

34. $3a - [4a - \{a - (6a - \overline{2a + 5})\} - (7a - \overline{3a + 13})].$

35. $4c - [3c - (5c - x) - \{2c - (4c + 3x) - (5c - \overline{x + 4})\}].$

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EXERCISES IN ALGEBRAIC EXPRESSION.

1. Write seven times the third power of x, diminished by three times the sum of $2a$ and $5b$.

2. If x represents an odd number, what will represent the next smaller odd number?

3. If A can do a piece of work in x days and B can do it in y days, what part of it can both do in one day?

4. Write four times the product of a square and x cube, increased by a times the quantity $2x - 3y$.

5. What is the interest on x dollars for a years at five per cent per annum?

6. In how many days can m men do as much work as n men can do in a days?

7. Write an expression for the sum of four consecutive numbers of which x is the largest.

8. If a man can perform a piece of work in five days, what part of it can he do in x days?

9. How many square yards are there in a floor which is x yards long and y feet wide?

10. If $2x - 1$ represents an odd number, what will represent the next larger odd number?

11. How many days will it take a man to build m yards of wall, if he builds n feet a day?

12. How long will it take a person to walk x miles, if he walks at the rate of 15 miles in a hours?

13. Write an expression for the sum of three consecutive odd numbers of which x is the smallest.

14. How many square feet are there in a piece of paper a yards long and x inches wide?

15. A has y sheep, B has 8 less than A, and C has 15 more than A and B together. How many have all?

PROBLEMS.

1. The greater of two numbers is seven times the less, and their sum is 184. What is the larger number?

2. A is three times as old as B and 8 years older than C. The sum of their ages is 90 years. Find C's age.

3. In a company of 98 persons it was found that there were twice as many women as men, and twice as many children as women. How many children were there?

4. The sum of two numbers is 240. If the smaller number were doubled, it would still be six less than the larger number. Find the larger number.

5. The sum of two numbers is 224, and their difference is six times the smaller number. Find the larger number.

6. After a farmer had sold one-sixth of his wheat, he still had four hundred thirty-two bushels more than he sold. How many bushels had he left?

7. The sum of two numbers is eighty-four, and three times the smaller number exceeds twice the larger number by twelve. Find the larger number.

8. A, B, and C together earn two thousand eight hundred dollars. A's salary is $200 more than B's, and $300 less than C's. Find C's salary.

9. A man rode 118 miles in three days, riding 6 miles more the second day than the first, and 4 miles more the third day than the second. How far did he ride the third day?

10. A boy bought oranges at four cents apiece and had 11 cents left; but if he had bought them at six cents apiece, he would have needed 21 cents more to pay for them. How many oranges did he buy?

11. A and B together earn $175 a month; A and C, $225; B and C, $315. How much do all earn?

12. The sum of three numbers is 142. The second exceeds the first by 9, and the third is 17 less than the second. Find the sum of the second and third.

13. Three men raised 1530 bushels of oats. A had three times as many bushels as C, and 248 bushels more than B. How many bushels did B raise?

14. A has three times as much money as B; but if A gives B six dollars, they will then have equal amounts. How much money have both?

15. Each of two boys has the same number of marbles. If John buys 35 more and Frank loses 15, John will then have twice as many as Frank. How many have both?

16. A boy has two dollars forty cents in dimes and 5-cent pieces. If he has six times as many 5-cent pieces as dimes, how many coins has he?

17. A has one-third as many sheep as B. If A should double his flock and B should sell eighty, they would then have the same number. How many has B?

18. A has one-fourth as much money as B, and C has $2\frac{3}{5}$ times as much as A and B. If C's money exceeds B's by three thousand six hundred dollars, how much have all?

19. Frank has one-fourth as many marbles as John. If John loses 206 and Frank loses 14, they will each have the same number left. How many marbles have both?

20. A man sold forty acres of land in two equal parts for three hundred sixty dollars, receiving twice as much per acre for the second as for the first. How much did he receive per acre for the second piece?

TO THE TEACHER. — Teach 74 . . . . 81.

MULTIPLICATION.

1.	**2.**	**3.**	**4.**	**5.**	**6.**	**7.**
$4a$	$-5x$	$-7c$	$-8b$	$6a$	$-7y$	$-5x$
$3a$	$-4x$	$5c$	$-6b$	$4a$	$-4y$	$4x$
8.	**9.**	**10.**	**11.**	**12.**	**13.**	**14.**
$5c^3$	$-6b$	$-7a^3$	$4x$	$3m^2$	$-4a^4$	$-8y$
$3c$	$9b^2$	$-5a^2$	$-3x$	$2m^2$	$-7a$	$2y^3$
15.	**16.**	**17.**	**18.**	**19.**	**20.**	**21.**
ab	$-ax$	$-ac$	xy	a^2b	$-ab$	$-a^2b$
ab	$-ay$	ac^2	$-xy^3$	ab^2	cd	$-ab^4$
22.	**23.**	**24.**	**25.**	**26.**	**27.**	**28.**
c^3d	$-am^3$	$-y^4$	x^2y	ac^2	$-6b^2$	$-ab$
cd^2	m^2	$-7y^2$	$-x^3y$	ab^2	$-5a^2$	xy
29.	**30.**	**31.**	**32.**	**33.**	**34.**	**35.**
$7a^3$	$-a^2x$	$-bc^3$	xy^2	a^2x	$-4c^2$	$-ab$
$5b^2$	$-a^3x$	ac	$-xz^2$	ax^2	$3d^2$	$-xy$
36.	**37.**	**38.**	**39.**	**40.**	**41.**	**42.**
$4a^m$	$-5c$	$-4x^m$	$6b^m$	ac^3	$-a^mx$	$-b^3c$
$3a$	$2c^m$	$-5x^n$	$-7b^2$	ac^n	$-ax^n$	bc^m
43.	**44.**	**45.**	**46.**	**47.**	**48.**	**49.**
$4c^2$	$-ab^m$	ab^n	$-x^ny$	$9a^m$	$-bc^n$	$-xy$
$6c^m$	$-ab^2$	$-b^2c$	xy^m	$5a^n$	b^mc	$-5x^n$
50.	**51.**	**52.**	**53.**	**54.**	**55.**	**56.**
a^{m+1}	$-c^{m-n}$	$-x^{m+3}$	b^{m+1}	a^{n+1}	$-x^n$	$-b^2$
a^{m-1}	c^n	$-x^{m-2}$	$-b^2$	a^n	$-x^{n+2}$	$-b^{n+2}$

To the Teacher. — Teach how to multiply a polynomial by a monomial. See 82.

Multiply:

1. $3x - y$ by $-2y$.
2. $ab + bc$ by ac.
3. $2x + y^2$ by $-4x$.
4. $4a - 5x$ by $3a$.
5. $4a - 3c$ by $-ac$.
6. $3x + 4y^2$ by $5x^2$.
7. $7b + 5y$ by $-4b$.
8. $2ac - 3c^2$ by a^2c^2.
9. $5x - 4y$ by $-5z$.
10. $3xy + 2y^2$ by x^2y^3.
11. $3b^2 + 5x^2$ by $-bx$.
12. $7ac - 5bc$ by $3bc^2$.
13. $6cd - c^2d^3$ by $-cd$.
14. $6am + 2bn$ by $4mn^2$.
15. $4ab^2 + 3a^2b$ by $-2ab$.
16. $5c^2d - 4cd^2$ by $3c^2d^2$.
17. $3a^2x - 5ax^2$ by $-3a^2x$.
18. $4ax^2 + 5a^2x$ by $6a^2x^2$.
19. $5x^2y^2 + 2x^2y$ by $-5ax^2$.
20. $7x^2y^3 - 2x^2y$ by $3x^2y^4$.
21. $7a^3b^2 - 4c^2d^3$ by $-abcd$.
22. $5a^3x^2 + 3b^2y^3$ by $abxy$.
23. $4a - c + 3x - 2y + xy$ by cx.
24. $3x + 2y - 5b + 6c - cx$ by by.
25. $4b + 2c - a + 3y - xy$ by $3b^2y$.
26. $ax - 3a + 2b - 5x + 4y$ by a^2y.
27. $5a - ab + 3b - 4c + 7b^2$ by $2a^2b$.
28. $b^2c^3 + 2b - 2c^2 + b^3c^2 - 4c$ by $3b^2c^2$.
29. $2c + 2c^2 - a^2c + 3a - a^2c^2$ by $4ac^3$.
30. $x^2y - 3x + 5y^2 - x^2y^2 + 3y$ by $2xy^2$.
31. $a^2b + ab^2 - 5a + 3b - 4b^2$ by $3a^2b^2$.
32. $3x - ax^2 + 4a - a^2x + 2x^2$ by $5a^2x^2$.
33. $3a^2 - a^3b^2 + 5b - b^2c^3 + 7a$ by $a^2b^2c^2$.
34. $x^2y^4 + 2z^2 - 4x + x^2z^2 - 5y$ by $x^4y^2z^2$.
35. $2x^2y - 3x^2y^3 + 4xy^2 - 3y + 4z$ by $4xy^2z$.
36. $3a^4 - 2a^3b + 3a^2b^2 - 2ab^3 + 2b^4$ by $4a^2b^3c^2$.

To the Teacher.—Teach how to multiply a polynomial by a polynomial. See 83.

Multiply:

1. $2x+y$ by $x+4y$.
2. $a-3x$ by $5a-x$.
3. $3b+2y$ by $b-y$.
4. $4a-c$ by $a+2c$.
5. $4a-3b$ by $3a-b$.
6. $3b+x$ by $2b-5x$.
7. $ac+xy$ by $ac+xy$.
8. $ab+cd$ by $ab-cd$.
9. $ax+cy$ by $ax-cy$.
10. $bx-dy$ by $bx+dy$.
11. $2x-5y$ by $3x+2y$.
12. $7a+5c$ by $3a-4c$.
13. $5a+4c$ by $3a-2c$.
14. $3c-5y$ by $6c+2y$.
15. $7x-5y$ by $2x-4y$.
16. $9a+2b$ by $2a+9b$.
17. $8b+3c$ by $5b-4c$.
18. $8x-3y$ by $4x-7y$.
19. $4x^2-2y^2$ by $5x+3y$.
20. $5a^2-4b^2$ by $3a-5b$.
21. $7x^2+5y^2$ by $2x-4y$.
22. $4c^2-6x^2$ by $5c-7x$.
23. $3a^3+5a-6$ by $4a^2-5$.
24. $5x^4-3x^3-2x^2$ by x^2-x.
25. $4x^3-2x^2+5x$ by $5x-3$.
26. $3c^2-2c+4$ by c^2-c+1.
27. $4a^2+5a-3$ by $3a^2-a+5$.
28. $3x^3+2x^2-4x+1$ by $5x+4$.
29. $x^3y-x^2y^2-3xy^3$ by $3x^2y-xy^2$.
30. $3ab+2b^2+a^2$ by $2a^2-3ab+b^2$.
31. $5ac-2a^2+3c^2$ by $3a^2+c^2-2ac$.
32. $3x^3-2x+3x^2-5$ by $3+2x^2-4x$.
33. $3xy^2+x^3-3x^2y-y^3$ by $y^2+2xy+x^2$.
34. $3m^2-2m+4-2m^3$ by $2m^2-4-3m$.
35. $2x^4+3x-2x^3-x^2-3$ by $3x+x^2+3$.
36. $3a^3+2a-4a^2+1$ by $2a^3-a-3a^2-4$.
37. $9x^3+2xy^2+3x^2y+4y^3$ by $4x^2+y^2-3xy$.

38. $4x^2-3x-2$ by x^2+x-3.

39. $3a^2-5a+3$ by a^2-4a+2.

40. $a^2+ab+2b^2$ by a^2-ab-b^2.

41. $x^2-3xy+5$ by $x^2+4xy-3$.

42. $b^2-4bc+c^2$ by $b^2-2bc-c^2$.

43. $a^4+b^4-a^2b^2$ by $a^4+b^4+a^2b^2$.

44. $3x^5+2x^2-4x^3-6$ by x^2-5.

45. $12x^2+4-3x$ by $6+5x^2-4x$.

46. $5x^2-12xy+22x^2y^2$ by $4y+3x$.

47. $2a+3+4a^2$ by $3a+4a^3-2a^2$.

48. $3x+1+x^2$ by $4x^2-2x+1-x^3$.

49. $2a^2-3b^2+4c^2$ by $6a^2+2b^2-4c^2$.

50. $a^3-3a^2c+5ac^2$ by $3a^2c-ac^2-6a^3$.

51. $a^4-a^2+3a-2a^3+1$ by a^2+1-a.

52. $3x^2-2xy+5y^2$ by $4x^2+2xy-3y^2$.

53. $3x^2+6y^2-4xy$ by $3x^2+6y^2+4xy$.

54. $2x^3-4x+2x^2+8$ by $3x^2-6-2x$.

55. $x^2+y^2+1-xy-x-y$ by $x+1+y$.

56. $5x^2+4-6x$ by $2x^2-4x+3-12x^3$.

57. $a^2+b^2+a-ab+b+1$ by $a-1+b$.

58. $a^2+b^2+c^2-ab-ca-cb$ by $a+b+c$.

59. $x^2+2xy+y^2+z^2$ by $x^2-2xy+y^2-z^2$.

60. $3x^2-x^4+4-x^3+2x$ by $3x^2+5-4x$.

61. $a^2-ac+c^2-bc+b^2+2ab$ by $a+b+c$.

62. $2m^3+3mn^2-4m^2n-n^3$ by m^2+n^2-mn.

Simplify the following:

1. $(a+x)(a-x)(a-x)(a+x)$.
2. $(1-x)(1+x)(1+x)(1+x)$.
3. $(1+a^4)(1+a^2)(1+a)(1-a)$.
4. $(x^2-2xy+y^2)(x^2+2xy+y^2)$.
5. $(a^4+x^4)(a^2+x^2)(a+x)(a-x)$.
6. $(a^2-c^2)(a^2-c^2)(a^2-c^2)(a^2-c^2)$.
7. $(a+c+x)(a+c-x)-(2ac-x^2)$.
8. $8(a-2c)(a+2c)-3(a-5c)^2+6a^2$.
9. $5(2a-x)^2-2(2a+2x)(2a-2x)-5x^2$.
10. $(a-3c)(c-2a)-(a-4c)(2c-a)+8ac$.
11. $(3x-2y)3(3x+2y)-3(4x+3y)^2-7xy$.
12. $6(x+2y)(x-2y)-(x-4y)^2+8(x^2+5y^2)$.
13. $(x-4)(x-2)-2x(x+4)+5(x+3)(x+4)$.
14. $(x+y)(y-z)-(z-u)(u+x)-(x+z)(y-u)$.
15. $(a-x)(x-y)+(x-y)(y-a)+(y-a)(a-x)$.

16. $(x^m+y^n)(x+y)$.
17. $(a-b)(a^m-b^m)$.
18. $(a^m-c^n)(a^m+c^n)$.
19. $(x^m+y^n)(x^n+y^m)$.
20. $(a^m+x^m)(a^n+x^n)$.
21. $(a^m-c^m)(a^m-c^m)$.
22. $(x^m-y^m)(x^{-1}+y^{-1})$.
23. $(m^{p-1}-n^{p-1})(m^2-n^2)$.
24. $(a^{2n}+c^{2n})(a^{-n}+c^{-n})$.
25. $(x^{m+1}-y^{m+1})(x^{-m}+y^{-m})$.
26. $(a^m-b^n)(a^{-2m}-b^{-2n})$.
27. $(x^{m+2}+y^{m+3})(x^{-2}-y^{-3})$.

28. $(4a^{3n}-2a^{2n}+a^n-1)(3a^n+1)$.
29. $(2a^{2m}-3a^mc^m+c^{2m})(2a^m+3c^m)$.
30. $(3x^m+2x^{m-1}-4x^{m-2}-1)(3x+1)$.
31. $(4a^n-3a^{n-1}+2a^{n-2}+3a^{n-3})(2a-1)$.

To the Teacher. — Teach 84 . . . . 90.

FORMULAS.

Write the following products by inspection:

1. $(a+c)(a+c)$.
2. $(x-y)(x-y)$.
3. $(b+c)(b-c)$.
4. $(a+b)(a+b)$.
5. $(a-x)(a-x)$.
6. $(a-x)(a+x)$.
7. $(e+c)(e+c)$.
8. $(x+y)(x-y)$.
9. $(c+d)(c+d)$.
10. $(a-d)(a-d)$.
11. $(b-x)(b+x)$.
12. $(x+y)(x+y)$.
13. $(a-b)(a-b)$.
14. $(c+d)(c-d)$.
15. $(r+s)(r+s)$.
16. $(b-c)(b-c)$.
17. $(a-b)(a+b)$.
18. $(a+x)(a+x)$.
19. $(b-x)(b-x)$.
20. $(e+c)(e-c)$.
21. $(b+c)(b+c)$.
22. $(a-c)(a-c)$.
23. $(a+1)(a-1)$.
24. $(b+8)(b+8)$.
25. $(b-4)(b-4)$.
26. $(x-2)(x+2)$.
27. $(a+7)(a+7)$.
28. $(3-x)(3-x)$.
29. $(c+3)(c-3)$.
30. $(x+6)(x+6)$.
31. $(2-a)(2-a)$.
32. $(b-4)(b+4)$.
33. $(c+5)(c+5)$.
34. $(1-c)(1-c)$.
35. $(a+5)(a-5)$.
36. $(x+4)(x+4)$.
37. $(a-7)(a-7)$.
38. $(x-6)(x+6)$.
39. $(a+3)(a+3)$.
40. $(c-6)(c-6)$.
41. $(c+7)(c-7)$.
42. $(c+1)(c+1)$.
43. $(5-a)(5-a)$.
44. $(b-8)(b+8)$.
45. $(2+b)(2+b)$.
46. $(1-x)(1+x)$.
47. $(x-9)(x-9)$.
48. $(x+7)(x-7)$.

49. $(2a - b)(2a - b)$.

50. $(a + 3b)(a - 3b)$.

51. $(4x + y)(4x + y)$.

52. $(a - 4x)(a + 4x)$.

53. $(a + 2x)(a - 2x)$.

54. $(3b + c)(3b + c)$.

55. $(x - 4y)(x - 4y)$.

56. $(5a - c)(5a + c)$.

57. $(a + 4b)(a + 4b)$.

58. $(3c - d)(3c - d)$.

59. $(2x + 3)(2x - 3)$.

60. $(1 + 3c)(1 + 3c)$.

61. $(4a - 6)(4a - 6)$.

62. $(3b - 1)(3b + 1)$.

63. $(2x + 7)(2x + 7)$.

64. $(1 - 9a)(1 - 9a)$.

65. $(5c + 3)(5c - 3)$.

66. $(4x + 4)(4x + 4)$.

67. $(2a - bc)(2a - bc)$.

68. $(ax - 3y)(ax + 3y)$.

69. $(ac + 4c)(ac + 4c)$.

70. $(3x - xy)(3x - xy)$.

71. $(8x + xy)(8x - xy)$.

72. $(bc + 5d)(bc + 5d)$.

73. $(3a - ab)(3a - ab)$.

74. $(ab - 7c)(ab + 7c)$.

75. $(6a + ac)(6a + ac)$.

76. $(xy - 4x)(xy - 4x)$.

77. $(8a - 9b)(8a + 9b)$.

78. $(2x + 3y)(2x + 3y)$.

79. $(4a - 2c)(4a - 2c)$.

80. $(7b + 6c)(7b - 6c)$.

81. $(5a + 3x)(5a + 3x)$.

82. $(4x - 5y)(4x - 5y)$.

83. $(9b - 7c)(9b + 7c)$.

84. $(2c + 5d)(2c + 5d)$.

85. $(2a - ac^2)(2a - ac^2)$.

86. $(x + 7xy^2)(x - 7xy^2)$.

87. $(5c + ac^2)(5c + ac^2)$.

88. $(b - 3ab^3)(b - 3ab^3)$.

89. $(9a - bc^3)(9a + bc^3)$.

90. $(x + 6yz^2)(x + 6yz^2)$.

91. $(7c - bc^3)(7c - bc^3)$.

92. $(8a + ax^2)(8a - ax^2)$.

93. $(a - 4ab^2)(a - 4ab^2)$.

94. $(8x - xy^2)(8x + xy^2)$.

95. $(3a^n + 2b^n)(3a^n + 2b^n)$.

96. $(5x^m - 2y^n)(5x^m - 2y^n)$.

97. $(6a^m - 7c^n)(6a^m + 7c^n)$.

98. $(4a^m + 3a^n)(4a^m + 3a^n)$.

99. $(7x^m - 2x^n)(7x^m - 2x^n)$.

100. $(2x^m + 7y^n)(2x^m - 7y^n)$.

TO THE TEACHER. — Teach 91 and 92,

Write the following products by inspection:

1. $(x+3)(x+2)$.
2. $(x-5)(x-2)$.
3. $(x+4)(x+1)$.
4. $(x-6)(x-3)$.
5. $(x+5)(x+4)$.
6. $(x-7)(x-2)$.
7. $(x+6)(x+1)$.
8. $(x-6)(x-4)$.
9. $(x+4)(x+2)$.
10. $(a-7)(a-8)$.
11. $(x+7)(x+3)$.
12. $(a-8)(a-1)$.
13. $(x+6)(x+5)$.
14. $(a-7)(a-5)$.
15. $(x+8)(x+4)$.
16. $(a-9)(a-1)$.
17. $(x+9)(x+6)$.
18. $(a-8)(a-3)$.
19. $(x+7)(x-4)$.
20. $(a+2)(a-6)$.
21. $(x-1)(x+6)$.
22. $(a-5)(a+4)$.
23. $(x+6)(x-5)$.
24. $(x+2)(x-3)$.
25. $(a-4)(a+5)$.
26. $(x-7)(x+3)$.
27. $(x+5)(x-1)$.
28. $(a+7)(a-8)$.
29. $(a-6)(a+7)$.
30. $(x-8)(x+5)$.
31. $(x+4)(x-2)$.
32. $(a+5)(a-9)$.
33. $(a-3)(a+4)$.
34. $(x-8)(x+6)$.
35. $(x+6)(x-2)$.
36. $(a+4)(a-9)$.
37. $(x+9)(x+2)$.
38. $(x-9)(x+2)$.
39. $(x-7)(x-4)$.
40. $(a+7)(a-2)$.
41. $(x+9)(x+5)$.
42. $(x+6)(x-9)$.
43. $(x-8)(x-3)$.
44. $(a-1)(a+7)$.
45. $(x+7)(x+9)$.
46. $(a-9)(a+7)$.
47. $(x-9)(x-4)$.
48. $(a+6)(a-4)$.
49. $(x+8)(x+2)$.
50. $(a+3)(a-9)$.

51. $(x+5)(x+1)$.

52. $(a-1)(a-7)$.

53. $(a-5)(a+2)$.

54. $(x-4)(x+9)$.

55. $(x+4)(x+3)$.

56. $(a+2)(a-6)$.

57. $(c-5)(c+7)$.

58. $(b-4)(b-5)$.

59. $(y+4)(y+8)$.

60. $(a+8)(a-7)$.

61. $(x-6)(x+5)$.

62. $(c-3)(c-1)$.

63. $(b+6)(b+2)$.

64. $(y+3)(y-4)$.

65. $(x-8)(x+9)$.

66. $(a-5)(a-3)$.

67. $(c+6)(c+7)$.

68. $(x-7)(x+6)$.

69. $(a+7)(a-3)$.

70. $(y-4)(y-3)$.

71. $(x^2+5)(x^2+3)$.

72. $(a^2+4)(a^2-9)$.

73. $(c^3+9)(c^3-2)$.

74. $(b^2-7)(b^2-1)$.

75. $(a^2+6)(a^2+1)$.

76. $(x^3-5)(x^3+4)$.

77. $(y^2-3)(y^2+8)$.

78. $(c^2-5)(c^2-3)$.

79. $(4x+2)(4x+3)$.

80. $(3a+5)(3a-6)$.

81. $(1+4x)(1+3x)$.

82. $(1-5a)(1+4a)$.

83. $(5c+6)(5c-2)$.

84. $(8y-3)(8y-2)$.

85. $(1-6c)(1+8c)$.

86. $(1-4b)(1-6b)$.

87. $(7b+4)(7b+1)$.

88. $(6x-9)(6x+3)$.

89. $(1+5y)(1+4y)$.

90. $(1+7x)(1-8x)$.

91. $(3a-3)(3a+8)$.

92. $(9c-4)(9c-1)$.

93. $(1+9a)(1-8a)$.

94. $(1-2c)(1-7c)$.

95. $(4x+6)(4x+4)$.

96. $(5y+6)(5y-7)$.

97. $(1+7x)(1+3x)$.

98. $(1-4a)(1+3a)$.

99. $(6b+7)(6b-5)$.

100. $(3x-3)(3x-4)$.

101. $(1-7y)(1+8y)$.

102. $(1-3x)(1-6x)$.

EXERCISES IN ALGEBRAIC EXPRESSION.

1. Write the third power of a, plus three times the product of b square multiplied by x, diminished by m times the square of the binomial $a-x$.

2. If $2x+1$ represents an odd number, what will represent the next smaller odd number?

3. At n cents a square foot, how much will it cost to plaster the ceiling of a room a yards long and b feet wide?

4. What is the interest on a dollars for eight years at x per cent per annum?

5. If A can do a piece of work in x days and B can do it in y days, what part of it can both do in 3 days?

6. If x represents an integer, does $2x+2$ represent an even or an odd number? Show why.

7. Write an expression for the sum of five consecutive numbers of which x is the middle one.

8. If it takes a men x days to do a piece of work, how many days will it take one man to do it?

9. If x represents the number of tens in a number and y the number of units, what will represent the number?

10. At n cents a square yard, how much will it cost to plaster the four walls of a room a feet long, b feet wide, and c feet high?

11. Write an expression for the sum of three consecutive even numbers of which x is the smallest.

12. While the minute-hand of a clock is passing over x spaces, how many spaces does the hour-hand pass over?

13. A merchant bought a pieces of silk at n cents a yard, and b pieces of another kind at m cents a yard. Express the cost in dollars, if each piece contained x yards.

14. At c dimes a square yard, how much will it cost in dollars to carpet a room a yards long and b feet wide?

PROBLEMS.

1. A has twice as much money as B, and B has twice as much as C. If they all have $455, how much has C?

2. A has twice as many sheep as B and 35 less than C. If all have 635, how many has A?

3. The sum of the ages of A, B, and C is one hundred forty-eight years. A is three times as old as C, and eight years younger than B. How old is B?

4. A horse, carriage, and harness cost $292. The horse cost ninety dollars more than the harness, and the carriage cost twenty-three dollars less than the horse. Find the cost of the horse and carriage.

5. A is three times as old as B, but ten years ago A was 5 times as old as B. Find A's age.

6. A lady spent $\frac{1}{8}$ of her money for a dress, when she found that she had left $144 more than she had spent. How much had she left?

7. Three men engage in business with a capital of $14,000. B invests one-half as much as A, and four hundred dollars more than C. How much has A invested?

8. Six boys and fifteen men earn two hundred sixty-four dollars a week. If each man earns four times as much as each boy, how much do the six boys earn per week?

9. A farmer sold his corn and wheat for $600; his corn and oats for $400; his wheat and oats for $520. How much did he receive for all his grain?

10. I bought 2 carriages for $305. At the same prices, 3 of the poorer ones would cost $15 more than 2 of the better ones. Find the cost of the better carriage.

11. There are three times as many pupils in one school as in another. If eighty pupils be transferred from the larger school to the smaller, the larger will still have twice as many as the other. How many pupils are there in both schools?

12. A man has three times as much money in the bank as he has in his safe. If he draws $175 from the bank and puts it into his safe, he will have the same amount in each place. How much has he in the bank?

13. The sum of the ages of mother and daughter is 60 years, and the difference between their ages is three times the daughter's age. Find the mother's age.

14. A harness cost one-third as much as a carriage, and a horse cost one and one-fourth times as much as harness and carriage. The horse cost forty-eight dollars more than the carriage. Find the cost of all.

15. A father and two sons earn $111 a month, the two sons receiving the same wages. If the sons' wages were doubled, they would receive only $3 less than their father. How much does the father earn per month?

16. A man bought two horses at the same price. He sold one at a profit of $45 and the other at a loss of $115, receiving twice as much for one as for the other. How much did both horses cost?

17. A farmer bought 45 sheep and had $30 left. If he had bought 60 sheep at the same price, he would have needed fifteen dollars more to pay for them. How much money had he?

18. A man paid a bill of $11.10 in quarters, dimes, and 5-cent pieces, giving three times as many dimes as 5-cent pieces, and twice as many quarters as dimes. How many coins did he give in payment?

TO THE TEACHER.—Teach 93 . . . . 99.

DIVISION.

1. $4\,ab^2)\,12\,a^3b^3$.	2. $3\,a^2b)-15\,a^4b^3$.	3. $-abc)-6\,ab^2c^3$.
4. $2\,bc^3)\,16\,b^2c^4$.	5. $5\,x^3y)-30\,x^5y^4$.	6. $-xyz)-5\,x^2yz^4$.
7. $3\,xy^2)\,18\,x^4y^4$.	8. $4\,a^2x)-28\,a^2x^2$.	9. $-acx)-7\,ac^3x^2$.
10. $5\,cx^3)\,15\,c^2x^4$.	11. $2\,b^3c)-18\,b^3c^3$.	12. $-bxy)-4\,b^2xy^4$.
13. $6\,b^2)\,18\,ab^3c^2$.	14. $7\,x)-21\,a^2x^2y^2$.	15. $-axy)-3\,ax^2y^2$.
16. $7\,ac^3)\,14\,a^3c^3$.	17. $6\,x^2y)-24\,x^3y^2$.	18. $-bcx)-2\,b^3cx^3$.
19. $4\,xy^4)\,16\,x^4y^4$.	20. $3\,a)-18\,a^3b^3c^2$.	21. $-abx)-6\,ab^3x^4$.
22. $3\,y^2)\,15\,xy^3z^2$.	23. $5\,a^2x)-25\,a^3x^4$.	24. $-xyz)-8\,x^3yz^2$.
25. $2\,ac^3)\,18\,a^2c^4$.	26. $4\,b)-20\,a^2b^3c^2$.	27. $-aby)-7\,ab^3y^2$.
28. $5\,x^2)\,10\,ax^2c^2$.	29. $2\,c^3d)-14\,c^4d^2$.	30. $-abx)-4\,a^4bx^3$.
31. $6\,xy^3)\,12\,x^3y^3$.	32. $7\,a)-28\,a^2b^2c^2$.	33. $-bxy)-3\,bx^3y^4$.
34. $7\,b^3)\,21\,ab^3c^2$.	35. $6\,x^3y)-24\,x^4y^2$.	36. $-acx)-2\,a^2cx^3$.
37. $4\,bc^2)\,20\,b^2c^3$.	38. $3\,d)-27\,a^2c^2d^2$.	39. $-xyz)-6\,xy^2z^2$.
40. $3\,d^3)\,27\,bc^2d^4$.	41. $5\,a^3x)-20\,a^3x^2$.	42. $-abx)-8\,a^2bx^2$.
43. $2\,ax^2)\,18\,a^2x^4$.	44. $4\,y)-16\,a^2x^2y^3$.	45. $-axy)-7\,ax^2y^2$.
46. $5\,c^3)\,25\,a^2bc^3$.	47. $2\,x^2y)-18\,x^3y^3$.	48. $-bcy)-4\,b^2cy^2$.
49. $6\,bx^2)\,36\,b^3x^3$.	50. $7\,z)-21\,x^2y^3z^2$.	51. $-xyz)-3\,xy^4z^4$.
52. $7\,y^2)\,42\,xy^3z^3$.	53. $6\,a^2y)-24\,a^2y^3$.	54. $-bxy)-2\,b^3xy^4$.
55. $4\,ax^3)\,20\,a^3x^4$.	56. $3\,b)-27\,a^2b^2c^2$.	57. $-acx)-6\,ac^4x^2$.
58. $3\,ab^2)\,15\,a^nb^n$.	59. $5\,b^2c)-35\,b^nc^n$.	60. $-bcx)-8\,b^ncx^n$.
61. $2\,x^2y)\,14\,x^ny^n$.	62. $4\,a^nx)-32\,a^2x^n$.	63. $-abc)-7\,ab^nc^n$.
64. $5\,b^nc)\,15\,b^2c^n$.	65. $2\,x^3y)-24\,x^ny^n$.	66. $-ad)-4\,a^nbcd^n$.
67. $6\,a^nx)\,36\,a^3x^n$.	68. $7\,b^nc)-35\,b^2c^n$.	69. $-by)-3\,ab^nxy^n$.
70. $7\,c^nx)\,35\,c^nx^2$.	71. $6\,a^3x)-42\,a^nx^n$.	72. $-xy)-2\,ax^ncy^n$.

To the Teacher. — Teach how to divide a polynomial by a monomial.

Divide:

1. $6\,ac^3 - 8\,a^2c + 10\,a^3c^3 - 4\,ac^2$ by $2\,ac$.
2. $8\,x^4y + 4\,x^3y^2 - 16\,x^3y + 4\,x^2y$ by $4\,x^2y$.
3. $6\,b^2c^3 + 10\,b^2c^4 - 8\,b^3c^2 - 2\,b^2c^2$ by $2\,bc^2$.
4. $9\,a^2b^3 - 3\,a^3b^4 + 15\,ab^3 - 6\,ab^2$ by $3\,ab^2$.
5. $8\,x^4y^2 + 4\,x^3y^2 - 16\,x^2y^3 + 8\,x^4y$ by $4\,x^2y$.
6. $5\,a^3c^4 + 10\,a^4c^2 - 5\,a^3c^3 + 5\,a^2c^2$ by $5\,a^2c^2$.
7. $4\,b^4c^3 + 8\,b^3c^4d - 16\,b^5c^3 + 12\,b^3c^2$ by $4\,b^3c^2$.
8. $12\,a^3x^4 + 3\,a^4x^4 - 6\,a^2x^3y - 11\,a^2x^3$ by a^2x^3.
9. $10\,x^4z^4 - 5\,x^3yz^5 + 15\,x^4z^5 - 20\,x^3z^4$ by $5\,x^2z^4$.
10. $18\,a^3c^3 + 12\,a^3bc^2 - 24\,a^4c^3 + 6\,a^3c^2$ by $6\,a^3c^2$.
11. $32\,ab^3c^3 - 16\,b^4c^3d + 16\,b^3c^4 - 8\,b^4c^2$ by $8\,b^2c^2$.
12. $14\,ab^4c^2 + 28\,b^4c^3d - 21\,b^3c^4 + 7\,b^3c^2$ by $7\,b^3c^2$.
13. $12\,b^4x^3y - 18\,b^3cx^5 + 15\,b^4x^4 - 6\,b^2x^3$ by $3\,b^2x^3$.
14. $15\,a^4c^3 - 10\,a^3bc^2d + 25\,a^3c^3 - 5\,a^4c^2$ by $5\,a^3c^2$.
15. $16\,x^3y^4z + 24\,ax^2y^3 - 20\,x^4y^2 + 4\,x^2y^2$ by $4\,x^2y^2$.
16. $36\,x^4y^3 + 18\,x^5y^4 - 24\,bx^4y^3z + 6\,x^4y^2$ by $6\,x^4y^2$.
17. $18\,b^nc^2d + 21\,ab^2c^n - 15\,b^{2n}c + 6\,b^nc^3$ by $3\,b^2c^n$.
18. $18\,a^4x^3 - 24\,a^3bcx^4 + 15\,a^3x^3 - 9\,a^2x^4$ by $3\,a^2x^3$.
19. $25\,a^3b^3c + 20\,a^4b^4d - 15\,a^2b^5 - 5\,a^2b^3$ by $5\,a^2b^3$.
20. $16\,a^nb^nc^2 - 24\,a^3b^nd + 20\,a^4b^2 - 8\,a^nb^3$ by $4\,a^3b$.
21. $14\,x^3y^n - 28\,ax^ny^nz + 21\,xy^{2n} - 7\,x^2y^n$ by $7\,x^ny^2$.
22. $8\,ab^nc^n + 6\,a^2b^nc^3 - 4\,a^{2n}b^nc^2 + 2\,a^nb^2c^n$ by $2\,a^nb^2c$.
23. $9\,x^3y^2z^n - 6\,x^2y^mz^n + 3\,x^my^2z^m - 3\,x^ny^nz^2$ by $3\,x^ny^nz$.
24. $6\,bc^nx^m + 8\,b^3c^mx^3 - 4\,b^mc^{2n}x^m - 2\,b^{2n}c^mx^n$ by $2\,b^mc^2x^n$.
25. $4\,a^2c^mx^n - 8\,a^{3m}c^{2n}x^m + 4\,a^nc^mx^{n-1} - 4\,acx^{2n-2}$ by $4\,a^mc^nx$.

To THE TEACHER. — Teach how to divide a polynomial by a polynomial.

Divide:

1. x^2+x-30 by $x+6$.
2. x^2-x-56 by $x-8$.
3. a^2-a-20 by $a-5$.
4. b^2+b-42 by $7+b$.
5. c^2-c-72 by $c-9$.
6. x^2+x-56 by $x+8$.
7. $20+x^2-9x$ by $x-4$.
8. $a^2+7a+10$ by $2+a$.
9. $b^2-4b-21$ by $3+b$.
10. $5y+y^2-84$ by $y-7$.
11. $x^2+6x-16$ by $x-2$.
12. $a^2-7a-78$ by $6+a$.
13. $x^2+13x+36$ by $x+4$.
14. $c^2+11c-12$ by $c-1$.
15. $14a+a^2-72$ by $a-4$.
16. $x^2-13x+12$ by $x-1$.
17. $b^2-17b-18$ by $b+1$.
18. $c^2-32c+60$ by $c-2$.
19. $15x^3+x^6+56$ by $7+x^3$.
20. a^4+15a^2+36 by a^2+3.
21. y^4-19y^2+48 by y^2-3.
22. b^6-15b^3+56 by b^3-7.
23. a^6+50a^3+96 by a^3+2.
24. y^4+47y^2-48 by y^2-1.
25. $a^4+ac^3+a^3c+c^4$ by $a+c$.
26. $b^3+b^2x+bx^2+x^3$ by $b+x$.
27. $a^3-ax^2-a^2x+x^3$ by $a-x$.
28. $a^4+c^4+a^2c^2$ by a^2+c^2+ac.
29. $ac^3-abc-bc^2+b^2$ by $ac-b$.
30. $x^3+3xy^2-3x^2y-y^3$ by $x-y$.
31. $x^2-y^2+z^2-2xz$ by $x+y-z$.
32. $a^2+2xy-y^2-x^2$ by $a+x-y$.
33. $32x^3-6x+1$ by $8x^2+2x-1$.
34. $4a^4-6a^2-8a^3+24$ by $2a-4$.
35. $4b^3+4ab^2-3a^2b-3a^3$ by $b+a$.
36. $16a^4+81x^4-72a^2x^2$ by $2a-3x$.
37. $25x^5-8x-2x^2-x^3$ by $5x^2-4x$.

38. a^3+1 by $a+1$.

39. x^5+1 by $x+1$.

40. x^3+8 by $x+2$.

41. $8-x^3$ by $2-x$.

42. a^4-b^4 by $a-b$.

43. x^5+y^5 by $x+y$.

44. x^6+y^6 by x^2+y^2.

45. $27-c^6$ by $3-c^2$.

46. x^3+8y^3 by $x+2y$.

47. a^8-y^8 by a^2-y^2.

48. $a^{10}+x^{10}$ by a^2+x^2.

49. $x^{12}-y^{12}$ by x^2-y^2.

50. $64x^3-125y^3$ by $4x-5y$.

51. $8a^{12}+729$ by $2a^4+9$.

52. $27x^3+343y^3$ by $3x+7y$.

53. $512a^6-216$ by $8a^2-6$.

54. $a^4-6ac-9a^2-c^2$ by a^2+c+3a.

55. $x^4+9y^4+2x^2y^2$ by x^2+3y^2-2xy.

56. $b^4+16c^4+4b^2c^2$ by b^2+4c^2+2bc.

57. $3cx+2ax+10ac+15c^2$ by $x+5c$.

58. $20ab-12ac-5b^2+3bc$ by $4a-b$.

59. $6x-9x^2-1+4x^4$ by $3x+2x^2-1$.

60 $9y^4+24+50y-67y^2$ by y^2-6+y.

61. $c^4-a^2c^2+2a^3c-a^4$ by c^2-ac+a^2.

62. $9-19a^2+2a^4$ by $2a^3-a+6a^2-3$.

63. $x^4+y^4+4xy^3+4x^3y+6x^2y^2$ by $x+y$.

64. $x^3+y^3+5xy^2+5x^2y$ by x^2+y^2+4xy.

65. $4a+6a^5+3a^2-4-11a^3$ by $3a^2-4$.

66. $y-2x-6y^3+54x^3-3x^2y$ by $2x-y$.

67. $a^4+6a^2x^2+x^4-4ax^3-4a^3x$ by $a-x$.

68. $a^3-a^2+a^6-a^4+2a-1$ by $a+a^2-1$.

69. $27ax^2-25a^3+20a^2x-18x^3$ by $6x-5a$.

70. $a^6-3a^2-1-6a^4$ by $-a-2a^2+a^3-1$.

71. $3a^4+5b^4+3a^2c^2-8a^2b^2-3b^2c^2$ by a^2-b^2.

72. x^5+32 by $x+2$.

73. a^3+64 by $a+4$.

74. a^6-64 by a^2-4.

75. x^5-32 by $x-2$.

76. x^7-128 by $x-2$.

77. x^5+243 by $x+3$.

78. $8x^3+1$ by $2x+1$.

79. $81a^4-1$ by $3a+1$.

80. $27x^3-1$ by $3x-1$.

81. $81x^4-1$ by $3x-1$.

82. a^4-81x^4 by $a-3x$.

83. $81x^4-y^4$ by $3x+y$.

84. $25x^2-16y^2$ by $5x+4y$.

85. $64a^3+8b^3$ by $4a+2b$.

86. $36b^2-81c^2$ by $6b-9c$.

87. $81x^4-16y^4$ by $3x+2y$.

88. $27x^3+8y^3$ by $3x+2y$.

89. $16a^2-64b^2$ by $4a-8b$.

90. $16a^4-81b^4$ by $2a-3b$.

91. $49x^2-25y^2$ by $7x+5y$.

92. $4x^5-x^3+27$ by $4x^2-3x^3+9-6x+2x^4$.

93. $a^4-71a-36a^2-3a^3-21$ by a^2-3-8a.

94. $6a^2c^2-4ac^3-4a^3c+a^4+c^4$ by $c^2-2ac+a^2$.

95. $x^4+6x^2y^2+4xy^3+4x^3y+y^4$ by y^2+x^2+2xy.

96. $x^4-4xy^3+y^4-4x^3y+6x^2y^2$ by $x^2-2xy+y^2$.

97. $x^2-3z^2+xy-2y^2+2xz+7yz$ by $x-y+3z$.

98. $51a+15a^3-18+6a^5$ by $7a-4a^2+2a^3-2$.

99. $6a^4-2a+23a^2-31a^3-48$ by $3a^2+6-5a$.

100. $ax-3x^2-ay+2a^2-y^2-4xy$ by $2a+y+3x$.

101. $a^5+6a^2-46a+7a^3+2a^4-120$ by $4a+a^2+5$.

102. $50m^2-32m+15+15m^4-32m^3$ by $5+3m^2-4m$.

103. $-ab-12c^2-6b^2+2ac+2a^2+17bc$ by $2a-4c+3b$.

104. $4x^2y^2+6xy^3+10x^3y+15x^4-3y^4$ by $3x^2-y^2+2xy$.

105. $13x+4x^2+6+x^5-6x^3-2x^4$ by $x^3+3x+3x^2+1$.

106. $a^3+y^3+8x^3-6axy$ by $a^2+y^2+4x^2-2xy-2ax-ay$.

107. $53x^2y^3-49y^5+2x^5-9x^4y-7x^3y^2$ by $2x^2-7y^2-5xy$.

TO THE TEACHER.—Teach 102 . . . . 109.

Determine what binomial or binomials, if any, will divide each of the following expressions, and give the quotient or quotients in each case:

1. $x^3 - y^3$.	**2.** $a^3 + 1$.	**3.** $1 + x^4$.
4. $x^5 + y^5$.	**5.** $x^7 - 1$.	**6.** $1 + x^5$.
7. $x^3 - y^6$.	**8.** $a^4 + 1$.	**9.** $x^5 - 1$.
10. $x^6 + y^6$.	**11.** $x^3 - 8$.	**12.** $8 + x^6$.
13. $x^3 - y^9$.	**14.** $8 + x^3$.	**15.** $x^9 - 8$.
16. $a^3 - 27$.	**17.** $x^9 + y^3$.	**18.** $a^4 + x^4$.
19. $x^3 + 64$.	**20.** $a^9 - b^6$.	**21.** $a^3 + x^6$.
22. $27 - x^6$.	**23.** $x^3 + 64$.	**24.** $b^6 - c^3$.
25. $a^5 + 32$.	**26.** $a^4 + 81$.	**27.** $x^9 + 64$.
28. $x^9 - 64$.	**29.** $27x^3 + 1$.	**30.** $x^5 - 32$.
31. $x^6 - 125$.	**32.** $216 - a^6$.	**33.** $1 - 64x^3$.
34. $x^3 + 216$.	**35.** $8a^9 - 27$.	**36.** $x^3 + 512$.
37. $8x^3 - 27$.	**38.** $x^9 + 343$.	**39.** $a^4 + 625$.
40. $a^3 - 216$.	**41.** $216 - x^6$.	**42.** $x^6 - 512$.
43. $343 - x^6$.	**44.** $a^3 + 343$.	**45.** $b^{12} - 125$.
46. $x^3 - 512$.	**47.** $512 + x^9$.	**48.** $216 - x^{15}$.
49. $8x^3 + 729$.	**50.** $x^5 - 243$.	**51.** $729 - 64x^3$.
52. $729 - 8a^6$.	**53.** $125 + x^9$.	**54.** $27a^3 - 343$.
55. $x^3 + 1000$.	**56.** $a^{10} - 243$.	**57.** $729x^6 + 64$.
58. $1000 - x^9$.	**59.** $125 - 8x^9$.	**60.** $343 - 27a^6$.
61. $8x^3 - 27y^3$.	**62.** $8x^3 + 125$.	**63.** $81x^4 - 256$.
64. $125x^3 + 8y^6$.	**65.** $64a^3 - 27$.	**66.** $16a^4 + 64x^4$.
67. $27a^3 - 512x^6$.	**68.** $729 + 27x^6$.	**69.** $729 - 512x^3$.
70. $512x^6 + 27y^3$.	**71.** $16x^4 - 81y^4$.	**72.** $216x^4 - 625$.
73. $64a^9 - 343x^6$.	**74.** $216x^3 + 512$.	**75.** $81a^4 - 10000$.

EXERCISES IN ALGEBRAIC EXPRESSION.

1. Write five times the square of $a - x$, diminished by the product of the binomials $x - 7$ and $x - 9$.

2. If a represents an integer, when does $a + 1$ represent an even number? When does it represent an odd number?

3. Write an expression for the sum of five consecutive even numbers of which x is the middle one.

4. If A can do a piece of work in 3 days and B can do it in 4 days, what part of it can both do in x days?

5. If x represents the number of hundreds in a number, y the number of tens, and z the number of units, what will represent the number?

6. Write an expression for the sum of four consecutive odd numbers of which x is the largest.

7. What is the interest on eight hundred forty dollars for a months at x per cent per annum?

8. At n cents a square yard, how much will it cost to plaster the walls and ceiling of a room a feet long, b feet wide, and c feet high?

9. If $2x - 5$ represents an odd number, what will represent the next smaller odd number?

10. What algebraic expression will represent the quotient of a number of three figures divided by three times the sum of its digits?

11. A room is x yards long, y feet wide, and a feet high. How many square yards of paper will cover the walls, allowing for 3 windows and 2 doors, each m feet long and n feet wide?

12. A man worked a weeks at b dollars a week, and his son worked c weeks at d dollars a week. They bought m tons of coal at n dollars a ton, and had y dollars left. Write the equation expressing these conditions.

PROBLEMS.

1. A man gave $110 to five boys, giving to each one six dollars more than to the next younger. How much did the oldest boy receive?

2. Three boys sold three hundred fifty-two papers. Frank sold twice as many as John and twenty-eight more than Harry. How many did Harry sell?

3. A man is 32 years older than his son. 12 years ago he was 5 times as old as his son. Find the father's age.

4. Three men invested $9600 in business. A put in $800 more than B, and C invested $400 less than A. How much did A and C together invest?

5. A farmer sold $\frac{8}{9}$ of his potatoes, when he found that he had left 868 bushels less than he sold. Find the value of his whole crop at 65 cents a bushel.

6. A, B, and C together earn $3450. A's salary is one-half of B's, and $450 less than C's. Find C's salary.

7. A lady paid fifty-six dollars for a hat and a dress. The difference in the cost was five times the cost of the hat. Find the cost of the dress.

8. A man paid $195 for his carriage and harness; $315 for his horse and carriage; and $230 for his harness and horse. How much did he pay for all?

9. A, B, and C have seven thousand two hundred fifty dollars. C has one-third as much as B, and B has $600 less than A. How much have A and B?

10. A merchant bought two pieces of silk for $70. At the same prices, four pieces of the cheaper kind would cost $16 more than two pieces of the better kind. Find the cost of the better piece.

11. Brown has one-fifth as many acres of land as Jones. If Jones sells Brown 140 acres, Jones will then have twice as many acres as Brown. How many acres have both?

12. A man paid $18,100 for two houses and a farm, paying the same sum for each house. If he had paid twice as much for each house, the two houses would have cost $4700 more than the farm. Find the cost of the farm.

13. James has one-third as many marbles as Frank. If James wins 125 and Frank loses 50, James will then have seven more than Frank. How many have both?

14. I bought two pieces of land at the same price. I sold one piece at a profit of $1600, and the other at a loss of $600, receiving twice as much for one piece as for the other. How much did each piece cost?

15. Three men form a partnership to engage in business. A invests one-half as much as C, and B invests $2\frac{2}{3}$ times as much as A and C. If B's investment exceeds C's by $15,000, what is the whole capital invested?

16. A collection of $14.88 consisted of a certain number of quarters, twice as many dimes as quarters, 3 times as many 5-cent pieces as dimes, half as many 2-cent pieces as 5-cent pieces, and as many pennies as all other coins. How many coins were there in the collection?

17. A lady bought silk at $3 a yard, and had $12 left. If she had bought twice as many yards at two dollars a yard, she would have needed four dollars more to pay for it. How much did she pay for the silk?

18. A farmer sold 20 lambs and 50 sheep for $240. He received twice as much per head for the sheep as for the lambs. How much did he receive for the 50 sheep?

TO THE TEACHER.—Teach 110 . . . . 114.

FACTORING.

CASE I.

1. $6ac - 3bc - 3c$.
2. $aby + bcy - bxy$.
3. $4xy - 6ax^2 - 8x$.
4. $2a^2x + 3ax - ax^2$.
5. $ab^2c - 4bc^2 - bcx$.
6. $9ac^2 + 5a^2c - ac$.
7. $4a^3x - 8ax^3 - 6ax$
8. $6x^2y + 9xy^2 - 3x$.
9. $8a^2b - 8a^3c + 4a^2x$.
10. $8x^2y - 6x^3 + 4xy$.
11. $9by^3 + 6b^2y - 3b^3y$.
12. $4a^2c + 8ac^2 - 4ac$.
13. $24x^3 - 12x^4 + 6x^2y^2$.
14. $6by^2 - 4b^2y + 2b^2y^2$.
15. $6a^3x^3 + 9a^2x^3 - 3ax^3y$.
16. $5x^3y^3 + 3x^2y - 4x^2y^2$.
17. $a^2b^2c^2 - ab^3c^2 + 2a^3b^2c$.
18. $3a^3x^3 + 2a^2x^4 - a^2x^2y^2$.
19. $9b^3c^3 - 6abc^2 - 3a^2bc^3$.
20. $ax^2y^3 + 3a^3xy^2 - a^2x^3y$.
21. $8a^2c^2 - 4a^2bc^2d - 4ac^2$.
22. $b^3cx^2 + 2b^2c^2x^2 - b^2cx^2$.
23. $6c^3d^2 + 9a^2c^2d^3 - 3c^3d^3$.
24. $a^3b^3c^3 - 3ab^4c^2 - a^2b^3c^2$.

TO THE TEACHER. — Teach 115.

1. $ac + ad + bc + bd$.
2. $ac - ax - bc + bx$.
3. $ax + ay + cx + cy$.
4. $bc - b - cn + n$.
5. $ax - ay + cx - cy$.
6. $cx + xy - cy - y^2$.
7. $m + n + cm + cn$.
8. $a^3 - 2a^2 - 3a + 6$.
9. $ac - a + 4c - 4$.
10. $m^2 + mn - m - n$.
11. $ab - ac - bc + c^2$.
12. $ab + bc + a + c$.
13. $a - ax + 3 - 3x$.
14. $ax + ay - cx - cy$.
15. $c^3 - c^2 + c - 1$.
16. $1 + c - c^2 - c^3$.
17. $ax + ay + dx + dy$.
18. $ax - cx - a + c$.
19. $a^3 - a^2c + ac^2 - c^3$.
20. $b^3 + 2b^2 - b - 2$.

21. $xy - x - y^2 + y.$

22. $x^2 + xy + 3x + 3y.$

23. $1 - a + a^2 - a^3.$

24. $x^3 + 5x^2 - 4x - 20.$

25. $a^2 + ac + ab + bc.$

26. $cm - my - cn + ny.$

27. $1 - c + 2c^2 - 2c^3.$

28. $x^2 + xy - x - y.$

29. $a^3 + a^2 + a + 1.$

30. $a^2 - ay - ax + xy.$

31. $2x^3 - 4x + 3x^2 - 6.$

32. $a^3 + a - a^2cd - cd.$

33. $x^3 + 4x^2 + 2x + 8.$

34. $3ax - 3ay - x + y.$

35. $ac^2 - acy + cy - y^2.$

36. $ax^2 - axy - bx + by.$

37. $2a^2 + 3ac + 2a + 3c.$

38. $ax - 3cx - 2ac + 6c^2.$

39. $adx + cd - anx - cn.$

40. $2ax - bx + 4ay - 2by.$

41. $c^3 + 5c^2 + 2c + 10.$

42. $b^3 - 7b^2 - 3b + 21.$

43. $x^5 - x^2y^2 + x^3y - y^3.$

44. $b^3 + bc^2 - b^2c - c^3.$

45. $a^4 - a^3x - ax^5 + x^6.$

46. $a^2b + ac^2 - abm - c^2m.$

To the Teacher.—Teach 116 . . . . 120.

Case II.

1. $x^4 + 2x^2y^2 + y^4.$

2. $x^2 + 4x + 4.$

3. $a^4 - 2a^2x + x^2.$

4. $a^2 - 2a + 1.$

5. $x^4 + 2x^2y + y^2.$

6. $c^4 + 1 + 2c^2.$

7. $4a^4 + 4x^4 - 8a^2x^2.$

8. $16 - 16c^3 + 4c^6.$

9. $9a^2 + 9c^2 + 18ac.$

10. $1 - 4a + 4a^2.$

11. $49x^2 + 42xy^2 + 9y^4.$

12. $\frac{1}{4}a^2c^2 - \frac{1}{3}ac + \frac{1}{9}.$

13. $4a^4x^2 + 4a^2x + 1.$

14. $36 + 9m^2 - 36m.$

15. $81x^4 - 90x^3 + 25x^2.$

16. $a^4 + 16b^2 + 8a^2b.$

17. $9a^4 + 16b^2 - 24a^2b.$

18. $20x^3 + 25x^4 + 4x^2.$

19. $4a^3x^3 + a^4x^2 + 4a^2x^4.$ **20.** $81x^6 + 16 - 72x^3.$

21. $64x^2 - 32x + 4.$ **22.** $\frac{1}{16}a^4 + 16x^2 + 2a^2x.$

23. $4x^4 - 4x^2y + y^2.$ **24.** $9a^2 + 6a + 1.$

25. $16x^4 + x^6 - 8x^5.$ **26.** $100 - 40a + 4a^2.$

27. $1 + 64m^4 + 16m^2.$ **28.** $9a^4b^2 - 30a^2b + 25.$

29. $\frac{4}{9}x^2 + xy + \frac{9}{16}y^2.$ **30.** $25a^6c^4 - 10a^3c^2 + 1.$

31. $81c^6 + c^4 + 18c^5.$ **32.** $49x^4 - 98x^2 + 49.$

33. $m^4 - 12m^3 + 36m^2.$ **34.** $121 + 4a^2c^2 + 44ac.$

35. $\frac{1}{4}x^6 + \frac{4}{9}x^4 - \frac{2}{3}x^5.$ **36.** $49a^2 - 14ac + c^2.$

37. $16y^2 + 16y^4 + 32y^3.$ **38.** $9m^2 + 4n^2 - 12mn.$

39. $1 + 18a^2x + 81a^4x^2.$ **40.** $64c^8 - 16c^7 + c^6.$

41. $\frac{9}{25}a^4 + \frac{1}{4}b^6 + \frac{3}{5}a^2b^3.$ **42.** $121b^2 - 88b + 16.$

43. $(a + x)^2 + 2(a + x) + 1.$

44. $8(x + y) + (x + y)^2 + 16.$

45. $(a + c)^2 - 16(a + c) + 64.$

46. $9(a - b)^2 - 12c(a - b) + 4c^2.$

47. $100 + 20ab + a^2b^2.$ **48.** $25x^4 - 50x^3 + 25x^2.$

49. $81a^4 + 4x^2 + 36a^2x.$ **50.** $\frac{1}{4}m^4 - \frac{1}{2}m^2n^2 + \frac{1}{4}n^4.$

51. $1 + 24ax + 144a^2x^2.$ **52.** $100x^6 + 16x^4 - 80x^5.$

53. $64 + 9a^4c^2 + 48a^2c.$ **54.** $49x^2 - 70xy + 25y^2.$

55. $\frac{1}{9}a^6c^4 - \frac{2}{9}a^5c^2 + \frac{1}{9}a^4.$ **56.** $121m^4 + 22m^2 + 1.$

57. $1 - 40x^4 + 400x^8.$ **58.** $36a^2 + 25b^2 + 60ab.$

59. $4x^2 - 20xy^2 + 25y^4.$ **60.** $25a^2 + 16c^2 + 40ac.$

61. $121 - 66m + 9m^2.$ **62.** $36a^8 + 36a^2 + 72a^5.$

63. $(x - y)^2 + 2ac(x - y) + a^2c^2.$

64. $25(a + x)^2 - 10c(a + x) + c^2.$

65. $9a^4(x - y)^2 + 48a^2(x - y) + 64.$

66. $(a+b)^2 - 4(a+b) + 4$.

67. $6(x-y) + (x-y)^2 + 9$.

68. $(x+y)^2 - 8(x+y) + 16$.

69. $(a+x)^2 + 25 + 10(a+x)$.

70. $12(x+y) + 4(x+y)^2 + 9$.

71. $a^2(b-c)^2 - 8a(b-c) + 16$.

72. $9(x-y)^2 - 12z(x-y) + 4z^2$.

73. $4a^4(a+b)^2 + 16 + 16a^2(a+b)$.

74. $9a^2(a-c)^2 - 12ab(a-c) + 4b^2$.

75. $16a^4(x+y)^2 + 9a^2 + 24a^3(x+y)$.

76. $9(2b-3x)^2 - 12a(2b-3x) + 4a^2$.

77. $(a+b)^2 + (c-d)^2 - 2(a+b)(c-d)$.

To the Teacher. — Teach 121 . . . . 123.

Case III.

1. $a^2 - c^2$.
2. $a^2 - 1$.
3. $1 - c^2$.
4. $a^2 - b^2$.
5. $b^2 - 1$.
6. $1 - b^2$.
7. $x^4 - y^4$.
8. $1 - x^4$.
9. $y^4 - 1$.
10. $a^6 - b^6$.
11. $x^6 - 1$.
12. $1 - y^6$.
13. $x^8 - y^8$.
14. $b^8 - 1$.
15. $1 - x^8$.
16. $m^4 - n^2$.
17. $x^6 - y^4$.
18. $a^2 - b^8$.
19. $4x^2 - 4$.
20. $9 - 4x^2$.
21. $9a^4 - 4$.
22. $9x^6 - y^2$.
23. $4x^8 - y^2$.
24. $a^6 - 9b^8$.
25. $9y^4 - 16$.
26. $25 - 9x^6$.
27. $4x^8 - 49$.
28. $4x^8 - 25$.
29. $64 - 9x^2$.
30. $9b^6 - 81$.
31. $9x^4 - 4y^2$.
32. $4a^6 - b^4c^2$.
33. $x^8y^2 - 9z^6$.

34. $9x^4-y^2$.
35. $4x^6-1$.
36. $1-9x^4$.
37. $4x^{2n}-y^{2n}$.
38. $9a^{2n}-1$.
39. $1-4x^{2n}$.
40. $4a^4x^4-y^6$.
41. $16-a^4b^2$.
42. $x^{4n}-25$.
43. $a^8b^4-c^2d^6$.
44. x^2y^6-49.
45. $36-x^{2n}$.
46. $a^6b^2-x^4y^8$.
47. $9x^4-y^6z^2$.
48. $x^{4n}-y^{2n}$.
49. $100x^4-y^2$.
50. $4a^8-b^2c^6$.
51. $x^{2n}-y^{4n}$.
52. a^4-144b^2.
53. $x^2y^2-9z^6$.
54. $x^{2m}-y^{2n}$.
55. $81x^6-9y^4$.
56. $36x^4-81$.
57. $x^{4m}-y^{4n}$.
58. $49x^4-4y^2$.
59. $16x^4-16$.
60. $x^{2m}-y^{2n}$.
61. $16x^4-16y^4$.
62. a^4-625b^4.
63. $9x^6-y^{4n}$.
64. $256x^8-y^8z^8$.
65. $81x^4-16y^4$.
66. $x^{4n}y^{4n}-16$.

TO THE TEACHER. — Teach 124.

1. $(x+y)^2-4$.
2. $(a+b)^2-(c+d)^2$.
3. $9-(a+c)^2$.
4. $(a+b)^2-(c-d)^2$.
5. $(x+y)^2-1$.
6. $(a-x)^2-(y+4)^2$.
7. $1-(b-c)^2$.
8. $(x-y)^2-(z-3)^2$.
9. $(b+c)^2-d^2$.
10. $(a+c)^2-(x+1)^2$.
11. $a^2-(b+c)^2$.
12. $(a+b)^2-(2c-d)^2$.
13. $(a-x)^2-c^2$.
14. $(x-2y)^2-(z-5)^2$.
15. $b^2-(a+c)^2$.
16. $(a-b)^2-(3c+4)^2$.
17. $(a-1)^2-d^2$.
18. $(2a+x)^2-(y-z)^2$.
19. $(a+x^2)^2-1$.
20. $(3a+c)^2-(2x+y)^2$.
21. $1-(x^2+y)^2$.
22. $(a-4b)^2-(c-7d)^2$.
23. $(x^2-3)^2-z^2$.
24. $(2x+3y)^2-(4z+5)^2$.
25. $b^2-(a-c^2)^2$.
26. $(2-3a)^2-(4b+6c)^2$.

27. $a^2 + 2a + 1 - b^2$.

28. $x^2 - y^2 + 2y - 1$.

29. $x^2 - 2x + 1 - y^2$.

30. $b^2 - 2c - c^2 - 1$.

31. $x^2 - y^2 + 4x + 4$.

32. $a^2 + 6x - x^2 - 9$.

33. $x^2 + y^2 - 2xy - 4$.

34. $9 - x^2 - y^2 - 2xy$.

35. $b^2 + c^2 - d^2 + 2bc$.

36. $a^2 - 2bc - b^2 - c^2$.

37. $(a^2 - b^2 - c^2)^2 - 4b^2c^2$.

38. $b^2 - a^2 - c^2 + 2ac$.

39. $4x^2y^2 - (x^2 + y^2 - z^2)^2$.

40. $a^2 + x^2 - c^2 - 2ax$.

41. $x^2 - y^2 - 2x + 1 + 2yz - z^2$.

42. $a^2 - c^2 + b^2 + 2ab - d^2 - 2cd$.

43. $a^2 - 2xy - 2ac - x^2 - y^2 + c^2$.

44. $a^2 + b^2 - x^2 - y^2 + 2ab - 2xy$.

45. $a^2 - c^2 - n^2 + m^2 - 2am + 2cn$.

46. $(a - b)(b^2 - c^2) - (b - c)(a^2 - b^2)$.

47. $4(2a - 3b)^4 - 9c^4$.

48. $x^2 + 1 - y^2 + 2x$.

49. $25a^4 - 9(3b - 2c)^4$.

50. $a^2 - 2x - x^2 - 1$.

51. $c^2 - d^2 - 2c + 1$.

52. $b^2 - d^2 + 2d - 1$.

53. $x^2 - z^2 - 6x + 9$.

54. $x^2 - 2y - y^2 - 1$.

55. $x^2 - 4 + y^2 + 2xy$.

56. $a^2 + 2ab + b^2 - 1$.

57. $a^2 + x^4 - 1 + 2ax^2$.

58. $1 - x^2 - y^2 - 2xy$.

59. $m^2 + n^2 - 1 + 2mn$.

60. $a^2 - b^2 - c^2 + 2bc$.

61. $x^2 + y^4 - z^4 - 2z^2 + 2xy^2 - 1$.

62. $a^4 - b^2 - 4c^2 + 9 - 6a^2 + 4bc$.

63. $a^2 - d^2 - y^2 - 2dy + c^2 + 2ac$.

64. $a^4 + c^2 - x^2 - y^4 - 2a^2c + 2xy^2$.

65. $4a^2 + 9b^2 - 9x^2 + 12ab - 4y^2 - 12xy$.

66. $(2a + 3b)^2 - 4c^2 - 3c(2a + 3b - 2c)$.

67. $a^2-2b-1-b^2$.
68. x^2-y^2+2x+1.
69. $1-b^2-c^2+2bc$.
70. $x^2+y^2-1-2xy$.
71. $1-a^2-b^2-2ab$.
72. $2ab-a^2-b^2+1$.
73. a^2-4b^2-4a+4.
74. $9-9x^2-6xy-y^2$.
75. $4ab-a^2+1-4b^2$.
76. $a^4-c^4+b^6-2a^2b^3$.
77. $a^2+16c^2-16+8ac$.
78. $8ac-4a^2-4c^2+4$.
79. $a^2-b^2+2b-1-2ac+c^2$.
80. $b^4-c^2+2b^2-2cd^2+1-d^4$.
81. $a^2+4c^2-9x^2-6x-1+4ac$.
82. $4a^2-4b^2-c^2+4bc-4a+1$.
83. $c^4-b^2-4x^2-6ac^2+4bx+9a^2$.

TO THE TEACHER. — Teach 125 . . . . 127.

CASE IV.

1. x^3-y^3.
2. a^3-1.
3. $1-b^3$.
4. x^5-y^5.
5. a^5-1.
6. $1-b^5$.
7. x^7-y^7.
8. a^7-1.
9. $1-b^7$.
10. x^3-y^6.
11. a^3-8.
12. $8-b^3$.
13. x^3-y^9.
14. a^6-8.
15. $8-b^9$.
16. x^3-27.
17. a^9-b^3.
18. b^6-c^3.
19. $27-x^6$.
20. a^9-b^6.
21. b^3-c^{12}.
22. x^9-64.
23. a^3-64.
24. a^3b^6-8.
25. x^9-27.
26. a^6-27.
27. $125-b^3$.
28. x^6-125.
29. $27-a^3$.
30. b^3-216.
31. $8x^3-y^{12}$.
32. $8a^9-27$.
33. $b^{12}-125$.

34. $a^5 - x^5$. 35. $x^5 - 1$. 36. $1 - y^5$.

37. $x^9 - y^3$. 38. $8 - x^3$. 39. $y^3 - 8$.

40. $x^9 - y^6$. 41. $x^6 - y^9$. 42. $27 - x^9$.

43. $8x^3 - 27$. 44. $27a^3 - 1$. 45. $x^5 - 32$.

46. $x^3 - 216$. 47. $216 - a^6$. 48. $1 - 64x^3$.

49. $343 - x^6$. 50. $a^9 - 343$. 51. $x^9 - 216$.

52. $x^3 - 512$. 53. $512 - x^9$. 54. $343 - x^3$.

55. $8x^3 - 729$. 56. $x^5 - 243$. 57. $a^6 - 512$.

58. $x^6 - 1000$. 59. $125 - 8x^9$. 60. $729 - 64x^3$.

61. $8x^3 - 125y^3$. 62. $64x^3 - 729$. 63. $27a^3 - 343$.

64. $512a^3 - 27b^6$. 65. $216x^6 - 125y^3$. 66. $729a^3 - 512$.

To the Teacher. — Teach 128 . . . . 130.

Case V.

1. $x^3 + y^3$. 2. $a^3 + 1$. 3. $1 + c^3$.

4. $x^5 + y^5$. 5. $1 + a^5$. 6. $b^5 + 1$.

7. $x^6 + y^6$. 8. $a^6 + 1$. 9. $1 + b^6$.

10. $x^7 + y^7$. 11. $1 + a^7$. 12. $b^7 + 1$.

13. $x^9 + y^9$. 14. $a^3 + 8$. 15. $8 + b^6$.

16. $x^3 + 64$. 17. $x^9 + y^3$. 18. $a^3 + x^3$.

19. $x^5 + 32$. 20. $27 + a^6$. 21. $b^9 + 64$.

22. $x^6 + 216$. 23. $343 + a^3$. 24. $x^3 + 512$.

25. $8x^3 + 729$. 26. $a^{10} + 243$. 27. $x^{12} + 216$.

28. $x^3 + 1000$. 29. $8x^6 + 125$. 30. $64x^3 + 729$.

31. $125x^3 + 8y^6$. 32. $27x^6 + 729$. 33. $343 + 27a^6$.

34. $512x^3 + 27y^6$. 35. $216a^3 + 125b^6$. 36. $729x^3 + 512$.

To the Teacher. — Teach 131 . . . . 133.

Case VI.

1. $x^2 + 7x + 12.$
2. $x^2 - 6x + 5.$
3. $x^2 + 7x - 18.$
4. $a^2 - 2a - 8.$
5. $x^2 + 8x + 15.$
6. $a^2 + 8a - 9.$
7. $x^2 - 9x + 20.$
8. $x^2 + x - 30.$
9. $x^2 + x - 132.$
10. $x^2 - x - 56.$
11. $x^2 + 13x + 36.$
12. $a^2 + a - 42.$
13. $a^2 + 17a + 30.$
14. $x^2 - 7x + 6.$
15. $a^2 - 14a + 48.$
16. $1 + 5x + 6x^2.$
17. $x^2 + 11x - 12.$
18. $1 - 8x - 9x^2.$
19. $a^2 + 11a + 30.$
20. $x^4 - 9x^2 + 8.$
21. $x^2 - 13x + 12.$
22. $a^4 - a^2 - 20.$
23. $a^2 - 17a - 18.$
24. $x^4 - 6x^2 - 72.$
25. $y^2 - 13y + 36.$
26. $x^4 - 4x^2 - 21.$
27. $x^2 - 4x - 320.$
28. $y^6 + 5y^3 - 84.$
29. $x^2 - 32x + 60.$
30. $a^2 + 9ac + 8c^2.$
31. $a^4 + 16a^2 + 48.$
32. $1 + 6a - 72a^2.$
33. $x^8 + 14x^4 - 72.$
34. $1 - 2a - 48a^2.$
35. $x^6 - 5x^3 - 300.$
36. $1 - 8a + 12a^2.$
37. $a^4 + 15a^2 + 36.$
38. $1 + 7x - 18x^2.$
39. $a^4 - 19a^2 + 48.$
40. $x^2 + 15x + 56.$
41. $x^4 - 16x^2 + 48.$
42. $a^2 + 15a - 16.$
43. $a^4 + 11a^2 - 42.$
44. $x^2 - 16x - 17.$
45. $x^4 - 14x^2 - 32.$
46. $a^2 + 13a + 12.$
47. $a^6 - 15a^3 + 56.$
48. $x^2 - 21x + 80.$
49. $x^2 + 23x + 102.$
50. $x^2 + 16x - 80.$

51. $x^2 + 5x - 6.$
52. $x^2 + x - 12.$
53. $x^2 - x - 132.$
54. $a^2 + a - 90.$
55. $a^2 + 2a - 15.$
56. $x^2 - x - 56.$
57. $y^2 - y - 110.$
58. $b^2 - b - 30.$
59. $x^2 + 8x - 48.$
60. $x^6 + x^3 - 72.$
61. $a^2 + 4a - 96.$
62. $1 - 9x + 8x^2.$
63. $x^2 + 17x + 16.$
64. $a^2 - ay - 2y^2.$
65. $b^2 + 15b + 14.$
66. $y^6 - 10y^3 + 9.$
67. $x^2 - 28x - 29.$
68. $1 + 7a + 12a^2.$
69. $a^2 + 12a - 28.$
70. $14x^2 + 9x + 1.$
71. $x^2 - 11x + 24.$
72. $1 + 8x + 12x^2.$
73. $y^2 + 47y - 48.$
74. $x^2 + 25x + 24.$
75. $x^2 - 19x + 84.$
76. $c^6 - 10c^3 - 11.$
77. $b^2 - 17b + 16.$
78. $y^4 - 14y^2 + 40.$
79. $x^2 + 13x + 12.$
80. $x^6 + 27x^3 + 72.$
81. $a^2 + 19a - 20.$
82. $y^4 - 15y^2 + 44.$
83. $x^2 + 23x - 24.$
84. $a^4 + 22a^2 + 96.$
85. $b^2 - 45b - 46.$
86. $1 + 21x - 72x^2.$
87. $a^4 + 16a^2 + 28.$
88. $x^2 - 21x - 100.$
89. $x^4 + 50x^2 + 96.$
90. $y^2 - 26y + 120.$
91. $1 + 3x - 180x^2.$
92. $1 + 11x + 18x^2.$
93. $1 + 12a - 64a^2.$
94. $x^2 + 6ax - 16a^2.$
95. $x^4 - 36x^2 + 320.$
96. $1 - 19c^2 - 20c^4.$
97. $y^4 - 14y^2 - 147.$
98. $a^6 - 32a^3 + 192.$
99. $1 - 8a^2 - 105a^4.$
100. $x^4 - 72x^2 + 512.$
101. $x^2 + 37xy + 36y^2.$
102. $1 - 35a + 300a^2.$

TO THE TEACHER. — Teach 131.

1. $8x^2+x-9$.
2. $7x^2-x-6$.
3. $3x^2-x-2$.
4. $6x^2-x-7$.
5. $6x^2-x-2$.
6. $4x^2+x-5$.
7. $6x^2-x-5$.
8. $7x^2-x-8$.
9. $8a^2+6a-9$.
10. $5x^2-8x+3$.
11. $6x^2+2x-4$.
12. $6a^2-2a-8$.
13. $6x^2+7x+2$.
14. $2x^2+5x+3$.
15. $3x^2+6x+3$.
16. $3x^2+7x-6$.
17. $4x^2+3x-10$.
18. $8x^2-6x-35$.
19. $6x^2+17x+12$.
20. $8a^2+ax-9x^2$.
21. $8x^2+45x-18$.
22. $15b^2-19b-8$.
23. $12x^2+19x+7$.
24. $48a^2-46a-9$.
25. $20b^2-4b-16$.
26. $10x^2-23x-5$.
27. $9a^2-32a-16$.
28. $6x^2+59x-10$.
29. $4x^2+8xy+3y^2$.
30. $10b^2-89b-9$.
31. $12c^2-17c-14$.
32. $18a^2+ax-4x^2$.
33. $12y^2+31y-15$.
34. $12x^2+13x-14$.
35. $14x^2+45x-14$.
36. $12a^2-11a-36$.
37. $9a^2+18ab+8b^2$.
38. $10x^2+19x-15$.
39. $9a^2+8ax-20x^2$.
40. $8a^2-34ac-9c^2$.
41. $9x^2-31xy+12y^2$.
42. $15b^2-4bc-4c^2$.
43. $5a^2+53ac-22c^2$.
44. $8a^2+53ax-21x^2$.
45. $8a^2-97ax+12x^2$.
46. $50x^2+35xy-4y^2$.
47. $15x^2-22xy-9y^2$.
48. $8b^2-37bc-15c^2$.
49. $8b^2-49bx-49x^2$.
50. $12a^2+23ax-9x^2$.
51. $10x^2-29xy+10y^2$.
52. $6x^2-23xy+21y^2$.

TO THE TEACHER.—Teach 135 . . . . 138.

CASE VII.

1. $9a^4 - 15a^2 + 1.$
2. $4a^4 - 13a^2 + 1.$
3. $x^4 + 2x^2y^2 + 9y^4.$
4. $4a^4 - 5a^2b^2 + b^4.$
5. $4a^4 - 53a^2x^2 + x^4.$
6. $9b^4 + 3b^2c^2 + 4c^4.$
7. $9x^4 + 8x^2y^2 + 16y^4.$
8. $a^4 - 26a^2c^2 + 25c^4.$
9. $49x^8 - 50x^4y^4 + y^8.$
10. $36x^4 + 11x^2y^2 + y^4.$
11. $25a^4 - 34a^2x^2 + 9x^4.$
12. $81a^4 - 45a^2c^2 + 4c^4.$
13. $49x^8 - 58x^4y^4 + 9y^8.$
14. $9x^4 - 52x^2y^2 + 64y^4.$
15. $16a^4 - 41a^2c^2 + 25c^4.$
16. $36a^4 + 35a^2x^2 + 25x^4.$
17. $121x^4 + 7x^2y^2 + 16y^4.$
18. $25b^4 - 89b^2x^2 + 64x^4.$
19. $16x^4 - 65x^2y^2 + 49y^4.$
20. $81x^8 + 41x^4y^4 + 25y^8.$
21. $25x^4 - 61x^2y^2 + 36y^4.$
22. $36x^8 - 85x^4y^4 + 49y^8.$
23. $49m^4 + 31m^2n^2 + 64n^4.$
24. $64x^8 + 87x^4y^4 + 49y^8.$

REVIEW OF FACTORING.

1. $a^2 + ac + ay + cy.$
2. $x^4 - x.$
3. $a^2 - m^4 + c^2 + 2ac.$
4. $20x^2 - 8x - 9.$
5. $3ax - 3ay - 2cx + 2cy.$
6. $5ac^4 + 5ac.$
7. $x^7y - 25x^4y^4 + 24xy^7.$
8. $x^4 - z^4 - 2x^2y + y^2.$
9. $(a-c)^2 - 10(a-c) + 25.$
10. $x^4 - 81.$
11. $25x^4 + 65x^2y^2 + 81y^4.$
12. $c + y + cx + xy.$
13. $y^8 - 19y^4z + 84z^2.$
14. $x^{12} + y^{12}.$
15. $(a+b)^2 + 2c(a+b) + c^2.$
16. $x^2 + y^2 - z^2 - 2xy.$
17. $a^4 - a^2c^2 - a^2x^2 + c^2x^2.$
18. $6a^2 + 11a - 72.$
19. $36a^4 + 116a^2x^2 + 121x^4.$
20. $x^8y + xy.$
21. $xy + nx + y + n.$
22. $8c^2 + 31c - 4.$
23. $x^2 - y^2 - n^2 - 2ny + m^2 + 2mx.$

24. $49x^4 + 59x^2y^2 + 100y^4$.

25. $a^8 - 16$.

26. $x^2y^2 - x^2 - 4y^2 + 4$.

27. $x^6 + 4x^3 - 32$.

28. $x^4 - 33x^2y + 272y^2$.

29. $a^2 - c^2 + 4 + 4a$.

30. $64x^4 + 44x^2y^2 + 25y^4$.

31. $c^2 - b^2 - 2ab - a^2$.

32. $(a - b)^2 + (a - b) - 2$.

33. $x^2 - ax - cx + ac$.

34. $ab^2 - 4a + 2b^2 - 8$.

35. $x - 8x^4$.

36. $9x^2 + 80xy - 9y^2$.

37. $x^{10} + y^{10}$.

38. $x^4 - 10x^2y^4 + 24y^8$.

39. $27x^3 - 1$.

40. $(a^2 + 5a)^2 + 12(a^2 + 5a) + 36$.

41. $x^2 - z^2 - 2x - y^2 + 1 + 2yz$.

42. $c^2 - y^2 - d^2 + 2dy - 2cx + x^2$.

43. $a^2 - n^2 - m^2 + 2ab + b^2 - 2mn$.

44. $ax - bx + ay - by + x^2 + y^2 + 2xy$.

45. $81x^4 - 184x^2y^2 + 100y^4$.

46. $7a^5c^2 + 7a^2c^5$.

47. $b^2 - d^2 - 10b + 25$.

48. $12x^2 - 37x - 10$.

49. $x^2 - xy - x + y$.

50. $5a^8 - 5$.

51. $x^3 - x^2y - xy^2 + y^3$.

52. $2c - c^2 + b^2 - 1$.

53. $m + 5 - m^3 - 5m^2$.

54. $x^4 + 12x^2y^2 - 64y^4$.

55. $ac + cx - ax - x^2$.

56. $2y^{13} + 16y$.

57. $121x^4 + 112x^2y^2 + 64y^4$.

58. $20x^2 + 21x - 54$.

59. $x^2z - x^2u - y^2z + uy^2$.

60. $1 + 243a^{10}$.

61. $(x^2 - 3x)^2 - 2(x^2 - 3x) - 8$.

62. $(a^2 - 2x)^2 + 6(a^2 - 2x) + 9$.

63. $2xy + c^2 + d^2 - x^2 - 2cd - y^2$.

64. $c^2 - n^2 + x^2 - 2bn - b^2 + 2cx$.

65. $2ax + 3bx + cx + 2ay + 3by + cy$.

66. $x^5y + 23x^3y^4 + 120xy^7$.

67. $2a^5 - 2a$.

68. $x^4 - 17x^2y^2 + 16y^4$.

69. $5x^3 - 40x^6$.

70. $a^2 + 5a - ac - 5c$.

71. $x + x^{13}$.

72. $2a^2b + ab^2 - b^3$.

73. $12a^2 - a - 11$.

74. $3a^3 - 2a^2c - 3ac^2 + 2c^3$.

75. $12a^2 - 27b^2$.

76. $(x^2 - 2x)^2 - 2(x^2 - 2x) - 3$.

77. $1 - y^2 - 2x + x^2$.

78. $12a^2 - 25ax - 22x^2$.

79. $18x^2 - 32y^2$.

80. $2m + mn - 2 - n$.

81. $40ax^4 + 5ax$.

82. $4x^4 - 45x^2y^4 + 81y^8$.

83. $1 - 13y^2 + 36y^4$.

84. $a^2 + b - 2ab - a + b^2$.

85. $x^2 + 2xy + y^2 + 4x + 4y$.

86. $8(a^2 - 2x) + (a^2 - 2x)^2 + 15$.

87. $(x^2 - 4x)^2 - 10(x^2 - 4x) + 25$.

88. $m^2 - 1 - n^2 + 2n$.

89. $a^6 - 64$.

90. $25x^4 - 26x^2y^2 + y^4$.

91. $m + 32m^6$.

92. $abx^2 + (a^2 + b^2)xy + aby^2$.

93. $9y^2 + 70y - 16$.

94. $(x + y)^2 - (x + y) - 12$.

95. $c^8 + 8c^5 + 15c^2$.

96. $121x^4 + 98x^2y^2 + 81y^4$.

97. $x^{11}y + x^2y^{13}$.

98. $a^2 - m^2 + an - mn$.

99. $2b - b^2 - 1 + a^2$.

100. $a^2x - (a^2 - b^2)y - b^2x$.

101. $64x^3 - 125$.

102. $abx^2 - xy(a^2 + b^2) + aby^2$.

103. $a^5 - 5a^3 - 36a$.

104. $abc^2 + abd^2 + (a^2 + b^2)cd$.

105. $a^4 - a^3c + ac^3 - c^4$.

106. $acx^2 - abx - bcx + b^2$.

107. $x^6 - 35x^3 - 36$.

108. $a^2 + 4c - 1 + b^2 - 2ab - 4c^2$.

109. $m^2 - b^4 - 4cm - 9n^2 + 6b^2n + 4c^2$.

110. $2ax - 3bx - 4cx - 2ay + 3by + 4cy$.

111. $16x^4 - 25x^2y^2 + 9y^4$.
112. $a^{15} + a$.
113. $9x^2 - 52xy - 12y^2$.
114. $14c^2 - 47c - 7$.
115. $(a-b)^2 - 12(a-b) + 36$.
116. $m^{16} - 625$.
117. $49a^4 - 53a^2x^2 + 4x^4$.
118. $a^2 - c^2 - a - c$.
119. $x^6 + 49x^3y^2 + 48y^4$.
120. $27x^{13}y + xy^4$.
121. $ax - ay + nx - ny$.
122. $15a^2 + 2a - 45$.
123. $a^4 - 23a^2c^2 + 112c^4$.
124. $27x^3 - 1331$.
125. $81x^4 - 214x^2y^2 + 121y^4$.
126. $3x^2 + 11x - 20$.
127. $3a + 3b + a^2 + 2ab + b^2$.
128. $ax + cy - x - ay - cx + y$.
129. $(x^2 - 3xy)^2 - 8y(x^2 - 3xy) - 48y^2$.
130. $4a^4 - 9d^2 + 12bd - 12a^2 - 4b^2 + 9$.
131. $(x^2 + 4xy)^2 - 24y(x^2 + 4xy) + 144y^2$.
132. $a^2 - b^2 - 2a + 1$.
133. $x^8y + xy^8$.
134. $a + m + a^2 - m^2$.
135. $a^{16} - c^4$.
136. $a^6c^2 + 15a^4c^4 - 16a^2c^6$.
137. $64a^{11}b + 2ab^8$.
138. $6a^3 - 4a^2 - 9a + 6$.
139. $c - 25c^6 + 24c^{11}$.
140. $(a-c)^2 + (a-c) - 6$.
141. $405x^9y^5 - 5xy$.
142. $2cx - x^2 - c^2 + 1$.
143. $8a^2 - 95a - 12$.
144. $100a^4 - 205a^2c^2 + 81c^4$.
145. $125a^{16}b + ab^4$.
146. $x + 62x^4 - 128x^7$.
147. $m^3 - 64n^6$.
148. $a^2 - 2ab + b^2 - ac + bc$.
149. $x^4 - 2x^2(2x-1) + (2x-1)^2$.
150. $(x^4 - 5x^2)^2 + 8(x^4 - 5x^2) + 16$.
151. $(x^4 - 10x^2)^2 + 18(x^4 - 10x^2) + 81$.
152. $2a^3 - 2ac^2 - 3a^2b + 3bc^2 - a^2 + c^2$.

153. $(2x-1)^2-(3x+1)^2.$

154. $c^5-c^4+c^3-c^2.$

155. $m^6-19m^3-216.$

156. $5a-80a^5.$

157. $(2a+3)^2-(3a-4)^2.$

158. $x^2-y^2-4x+4y.$

159. $(2a+3x)^2-(3a-4x)^2.$

160. $2x^{16}y-2xy^{11}.$

161. $b^4+b^3+b^2+b.$

162. $1000x^3-1331y^6.$

163. $(5x+2y)^2-(4x-3y)^2.$

164. $a^2-b^2-2a+2b.$

165. $(7a-3c)^2-(2a+c)^2.$

166. $am-an+m^2-n^2.$

167. $\frac{1}{4}a^4+\frac{1}{3}a^3+\frac{1}{9}a^2.$

168. $4a^2+9b^2+12ab.$

169. $5a^2-8ab+3b^2-5a+3b.$

170. $(a+x)^2-1-2x(a+x-1).$

171. $3x^2+6xy+3y^2-3xz-3yz.$

172. $(x^2-2xy)^2+2y^2(x^2-2xy)+y^4.$

173. $25+4a^2-4b^2+12bc-9c^2-20a.$

174. $(8a+5x)^2-(3a-2x)^2.$

175. $48x^9y-1875xy^5.$

176. $x^8-43x^5+42x^2.$

177. $m^6+m^3-72.$

178. $3c^3-c^2+3c-1.$

179. $x^{12}-y^9z^3.$

180. $(9a+3b)^2-(2a-5b)^2.$

181. $8a^3b^6+27x^9y^{12}.$

182. $9x^4-10x^2y^4+y^8.$

183. $16a^3-54b^{12}.$

184. $(x-6)^2-(x+y-6)^2.$

185. $32a^{12}-2x^8.$

186. $1-a+a^2-a^3.$

187. $5x^3y^2-5x^2y^5.$

188. $(x+y-z)^2-(x-y-z)^2.$

189. $\frac{4}{9}a^2+\frac{9}{16}-a.$

190. $ax+by-y^2-ay-bx+x^2.$

191. $a^2-b^2-c^2+2bc+a+b-c.$

192. $x^3-y^3-x(x^2-y^2)+y(x-y)^2.$

193. $(a-c)(c^2-x^2)-(c-x)(a^2-c^2).$

194. $a^4-2a^2(3ac-2c^2)+(3ac-2c^2)^2$

195. $16x^4 - 97x^2y^2 + 81y^4$.

196. $x^8 - 80x^4 - 81$.

197. $(x+y)^3 + 1$.

198. $x^{12} + x^3$.

199. $a^6 - 19a^3b^6 + 88b^{12}$.

200. $81a^4b - 375ab^7$.

201. $1 + (a+b)^3$.

202. $729 + 1000a^3$.

203. $(x-y)^3 - 1$.

204. $1 - (a-b)^3$.

205. $4x^4 - 9y^2 + 12xy - 4x^2$.

206. $x^4 + x^2 + 1$.

207. $9x^2 - 4y^2 - 3xz + 2yz$.

208. $8a^6 - b^9$.

209. $49a^4 - 65a^2x^2 + 16x^4$.

210. $(a+x)^4 - 1$.

211. $x^8 - 19x^4y^4 + 48y^8$.

212. $1 - (a-b)^4$.

213. $a^2 + a + 3x - 9x^2$.

214. $x^7 - x$.

215. $a^4 + a^2 - 36 - 2a^3$.

216. $1029a^3c - 3c^4$.

217. $36x^4 - 37x^2y^4 + y^8$.

218. $729x^6 + 125y^{12}z^3$.

219. $(a+x)^3 + (2a-x)^3$.

220. $343x^3 - 64y^{12}$.

221. $8ab - 4b^2 + 4 - 4a^2$.

222. $486x^6 + 2x$.

223. $1 - x^2 - y^2 - 2xy$.

224. $48m^4 - 1875n^4$.

225. $(2x-y)^3 - (x-y)^3$.

226. $c^2 - 7c^5 - 8c^8$.

227. $a^5c + 65a^3c^3 + 64ac^5$.

228. $8a^3 - (a-x)^3$.

229. $25b^4 - 45b^2c^2 + 4c^4$.

230. $8(a+c)^3 + x^3$.

231. $4a^2x^2 - (a^2 + x^2 - y^2)^2$.

232. $a^7b + ab^7$.

233. $a^4 - 2a^3 + a^2 - 4a + 4$.

234. $162a^5b - 512ab^5$.

235. $a^3 - x^3 - 3ax(a-x)$.

236. $x^4 - 4 - 2x^3 + x^2$.

237. $3ab(a+b) + a^3 + b^3$.

238. $a^2 - x^2 - 2a - 2x$.

239. $a^{2m} + b^{2n} - 2a^mb^n$.

240. $16a^{4n} - 81b^{4n}$.

241. $x^{2m} - x^my^n - 6y^{2n}$.

242. $8a^{3n} - 27b^{3n}$.

243. $4x^{4n} + 11x^{2n}y^{2n} + 9y^{4n}$.

244. $729a^{6n} + 1000b^{3n}$.

245. $4a^{2n} + 19a^nb^n - 5b^{2n}$.

246. $a^{2n} - x^{2n} - a^n - x^n$.

To the Teacher. — Teach 139 . . . . 143.

HIGHEST COMMON DIVISOR.

Find the highest common divisor of the following:

1. $4a^3b^2$, $6a^2bc$, $2a^4b^3c$, $8a^3b^4$, and $10a^2b^2c^2$.
2. $9x^2y^3$, $6xy^4z$, $12x^3y^2z$, $3x^4y^4$, and $15x^2y^3z^4$.
3. $8a^3c^3$, $4a^2bc^4$, $16a^4b^2c^2$, $12a^4c^3$, and $20a^3b^2c^3$.
4. $3a^2b^4c^3$, $3ab^3c^3$, $4a^3b^4c^2$, $5a^2b^3c^4$, and $7a^2b^4c^3d$.
5. $10x^3y^3z^3$, $5x^4y^3z$, $15x^5y^4z^3$, $5x^3y^3z^3$, and $25a^4y^2z$.
6. $12a^2b^2x^2$, $6a^3b^3x$, $18a^4b^2x^3$, $24ab^2x^3$, and $42a^2bx^4$.
7. $35a^3b^2y^3$, $49a^2b^3y^2$ $42b^2x^2y^3$, $77a^2by^3$, and $63ab^4y^4$.
8. $44a^2b^3x^2$, $77x^2y^2z^3$, $66b^3c^2x^2$, $88a^4xy^4$, and $99x^2y^3z^4$.
9. $72a^2c^2x^2$, $63ab^4c^3$, $54a^2c^2x^2y$, $45b^3cx^2$, and $27a^3b^3c^2$.
10. $49x^4y^3z^2$, $21x^3y^4z^2$, $35ac^2x^4y^2$, $56x^5y^4z^3$, and $42x^3y^2z^3$.
11. $35a^2b^3c^2$, $40a^3b^2x^3$, $30a^2x^3y^2z$, $45b^2x^2y^3$, and $21a^2b^2c^2x$.
12. $x + y$, $x^4 - y^4$, and $x^3 + y^3$.
13. $ac + ad - bc - bd$ and $a^4 - b^4$.
14. $x^2 - 6x + 9$ and $x^2 - 7x + 12$.
15. $x^2 - 4$, $x^2 + 6x + 8$, and $x^6 + 8$.
16. $x^2 - 4x - 5$ and $3x^2 - 11x - 20$.
17. $x^2 + 2x - 15$, $x^3 - 27$, and $x^2 - 6x + 9$.
18. $6ax^2 - 6ax$, $2abx - 2ab$, and $2ax - 2a$.
19. $18ax^2 - 2a$, $54ax^3 + 2a$, and $18ax + 6a$.
20. $12x^2 - 18x$, $48x^3 - 108x$, and $24x^4 - 81x$.
21. $49a^2 - 16$, $16 - 56a + 49a^2$, and $63a^2 - 36a$.
22. $2ab + 2ay - 3bx - 3xy$ and $9x^2 - 12ax + 4a^2$.

26. $ax^5 + ax^4 - ax^3 - ax^2$ and $x^4y - 3x^2y - 2xy$.

27. $3x^4y - x^3y - 2x^2y$ and $3x^4 + 5x^3 + 2x^2 + 3x + 2$.

28. $3x^4 - 5x^3 + 7x^2 - 3x - 2$ and $6x^3 + 5x^2 - 8x - 3$.

29. $2x^4 - 3x^3 - 3x^2 - 3x - 5$ and $6x^3 - 17x^2 + x + 10$.

30. $3a^4 + 6a^3x + 6a^2x^2 + 3ax^3 - 6x^4$ and $a^3 - 2ax^2 + x^3$.

31. $2x^7 - 7x^6 + 2x^5 + 2x^4 + 3x^3$ and $x^5 - 2x^4 - 4x^3 + 3x^2$.

32. $4x^4 - 2x^3 - 4x^2 - 4x - 2$ and $15x^4 - 15x^2 - 30x - 15$.

33. $x^4 - x^3y - 3x^2y^2 + xy^3 + 2y^4$ and $x^3 - 3x^2y + 3xy^2 - 2y^3$.

34. $2x^5 + x^4y + x^3y^2 - xy^4 + y^5$ and $3x^3 + 3x^2y + 2xy^2 + 2y^3$.

35. $a^7 + 2a^6x + 2a^5x^2 + 2a^4x^3 + 2a^3x^4 + a^2x^5$ and $2a^4 - a^3x - 3a^2x^2$.

To the Teacher.—Teach 149 . . . . 154.

LOWEST COMMON MULTIPLE.

Find the lowest common multiple of the following:

1. $2abc$, $6ab^2c$, $3a^2b^2c$, $4ab^3$, and a^2bc.
2. $3x^2yz$, $2xy^2z$, $6x^3yz^2$, $9xy$, and xy^2z.
3. $3a^3cx$, $4bcx$, $9a^2cx^2$, $2b^2c$, and ac^3x.
4. $4bcx^2$, $3cxy$, $8b^2x^3y$, $2b^2x^2$, and bcx.
5. $4abx^2$, $2b^2xy$, $5a^2x^2y$, $4ax^2$, and bxy.
6. $5ac^2x$, $2acx$, $3a^2c^2x$, $6c^3y^2$, and axy.
7. $6bc^2x^2$, $4c^2x^3y^2$, bc^3x^2, $16b^2x$, and $bcxy$.
8. $2bc^2x^3$, $4bc^3x^2$, $16b^2cx^3$, bx^4, and abc^2x.
9. $5cx^2y^2$, $3cx^2y^3$, $2b^3xy^2$, $6b^2c^2$, and $bcxy$.
10. $2a^2b^2c$, $5a^2b^3c$, $8a^3bc^2$, $4a^4b^2$, and $abcd$.
11. $4ab^2c^2$, $2bc^4x$, $8ab^3x^3$, $7ax^5$, and a^2b^2cx.

12. x^2-1 and x^3-1.

13. $1-x^2$ and $1+x^3$.

14. a^2-x^2 and a^2+ax.

15. x^2-1 and x^2-3x-4.

16. $4a^2-4b^2$ and $6a-6b$.

17. $5(x^4-y^4)$ and $2(x^2+y^2)$.

18. $x^2-10x+16$ and x^3-8.

19. x^2-y^2 and $x^2-2xy+y^2$.

20. $x^2+2xy+y^2$ and x^2-y^2.

21. x^2+1, x^2-1, and x^4-1.

22. x^2+x, x^2-1, and x^3+1.

23. x^2-16 and $x^2+14x+40$.

24. $2a(a^2-c^2)$ and $3c(a-c)^2$.

25. $6+2x$, x^2-9, and $27+x^3$.

26. x^2-x-20 and x^2+x-12.

27. a^2-2a+1 and a^2-5a+4.

28. a^2+5a+6 and a^2+6a+8.

29. $x^2-6x-16$ and $x^2-11x+24$.

30. $4x^2-16$, $6x-12$, and $8x^3-64$.

31. $x^2+8ax+15a^2$ and $x^2-5ax-24a^2$.

32. $ax-4a+bx-4b$ and $a^2-3ab-4b^2$.

33. $2a^2+2ab$, $3ab-3b^2$, and $4a^2c-4b^2c$.

34. x^2-16, $x^2-8x+16$, and $x^2+8x+16$.

35. $2x^2-2$, $4x-4$, $8x+8$, and $12x^2+12$.

36. x^3-2x^2+4x-8 and x^3+2x^2-4x-8.

37. $x^3+x^2y-xy^2-y^3$ and $x^3-x^2y+xy^2-y^3$.

38. $x^2 + 4 - 5x$ and $3 - 4x + x^2$.

39. $12 - 7x + x^2$ and $x^2 + x - 20$.

40. $c^2 + 4c + 4$, $c^2 - 4$, and $c^4 - 16$.

41. $a^4 + 2a^3$, $a^2 - 4a + 4$, and $a^2 - 4$.

42. $1 - 2x + x^2$, $1 + 2x + x^2$, and $x^2 - 1$.

43. $2a(x^2 + y^2)$, $3c(x + y)$, and $ac(x - y)$.

44. $x^4 + 4x^2 + 4$, $4 - 4x^2 + x^4$, and $x^4 - 4$.

45. $5a^3(a^2 - x)$, $3a^2(a^2 + x)$, and $2a^4(a^4 + x^2)$.

46. $x^2 + 2x + 1$, $9 - 6x + x^2$, and $x^2 - 2x - 3$.

47. $a^2 - 4$, $a^2 - a - 6$, and $a^3 - 3a^2 - 4a + 12$.

48. $x^2 + 2x - 15$, $x^2 - 9x + 18$, and $x^2 - x - 30$.

49. $(a - b)(a - c)$, $(a - c)(b - c)$, and $(a - b)(b - c)$.

50. $a(a + b)(b + c)$, $b(a + c)(b + c)$, and $c(a + b)(a + c)$.

To THE TEACHER. — Teach 155.

Find the lowest common multiple of the following:

1. $x^3 + 3x^2 + 4x + 2$ and $2x^3 + x^2 + 1$.

2. $2a^4 - 2a^3 - 2a - 2$ and $a^4 - 2a^3 + a$.

3. $x^6 - 4x^5 + 56x - 15$ and $x^3 - 15x + 4$.

4. $6x^3 + 10x^2 + 8x + 4$ and $6x^3 - 2x^2 - 4$.

5. $x^4 - 11x^2 + 25$ and $x^4 - x^3 - 6x^2 + x + 5$.

6. $6x^3 - x^2 - 7x - 2$ and $6x^3 + 5x^2 - 5x - 2$.

7. $2x^3 - 5x^2 - 20x + 9$ and $2x^3 + x^2 - 43x - 9$.

8. $2x^3 + 5x^2 - 8x - 15$ and $4x^3 - 4x^2 - 9x + 5$.

9. $x^3 - 10x^2 + 33x - 36$ and $x^3 - 2x^2 - 23x + 60$.

10. $6x^3 - 8x^2 - 17x - 6$ and $12x^3 - x^2 - 21x - 10$.

11. $3x^3 + 17x^2 - 44x - 28$ and $6x^3 - 5x^2 - 33x + 28$.

To THE TEACHER. — Teach 156 . . . . 174.

FRACTIONS.

Reduce the following fractions to their lowest terms:

1. $\dfrac{4a^2b^2c}{8a^3b^2c}$.

2. $\dfrac{15b^2x^3y^3}{21b^3x^2y^3}$.

3. $\dfrac{3a^3b^2x^3y}{6a^4b^2x^2y}$.

4. $\dfrac{3a^3b^2c}{9a^2bc^2}$.

5. $\dfrac{18b^4x^2y^3}{27b^3x^3y^2}$.

6. $\dfrac{8a^2b^3xy^3}{4ab^2x^2y^2}$.

7. $\dfrac{6a^2b^4c}{8a^3b^2c}$.

8. $\dfrac{32x^2y^3z^2}{24b^2x^2y^3}$.

9. $\dfrac{4b^4x^2y^3z}{6ab^3x^3y^3}$.

10. $\dfrac{6a^3cx^2}{9a^2b^2x}$.

11. $\dfrac{23a^3x^4y^2}{46x^3y^2z^3}$.

12. $\dfrac{2a^3b^2xy^3}{3b^2xy^3z^2}$.

13. $\dfrac{2a^3b^2c}{4a^2b^3c}$.

14. $\dfrac{38b^3x^2y^4}{19b^4x^3y^3}$.

15. $\dfrac{4a^2b^3xy^3}{2a^3b^3x^2y}$.

16. $\dfrac{3a^3cx^3}{5a^2c^2x}$.

17. $\dfrac{35x^3y^2z^3}{42b^2x^3y^2}$.

18. $\dfrac{3a^3b^3x^3y}{9b^2x^4yz^2}$.

19. $\dfrac{3a^2b^3c}{9a^3bc^2}$.

20. $\dfrac{34x^2y^2z^3}{17x^3y^2z^4}$.

21. $\dfrac{6bx^2y^2z^2}{3a^2bx^2y^2}$.

22. $\dfrac{2ac^2x^2}{6a^2b^2c}$.

23. $\dfrac{25b^2x^3y^4}{30b^2x^2y^3}$.

24. $\dfrac{4a^3bx^2y^3}{7bx^3y^3z^2}$.

25. $\dfrac{8a^3b^2c}{2a^2cx^2}$.

26. $\dfrac{21a^2x^3y^3}{42x^2y^4z^3}$.

27. $\dfrac{9b^2x^4y^3z}{3ab^3x^3y^3}$.

28. $\dfrac{6a^2cx^2}{3a^3c^2x}$.

29. $\dfrac{46a^3b^2x^4}{23b^2x^3y^2}$.

30. $\dfrac{6a^3b^4x^2y}{8a^2b^3x^2y}$.

31. $\dfrac{9a^3b^3c}{6a^2b^2c}$.

32. $\dfrac{16a^2b^3x^2}{24b^2x^3y^3}$.

33. $\dfrac{6b^2x^3yz^2}{9b^3x^4yz^3}$.

34. $\dfrac{8ac^2x^3}{6a^2c^3x}$.

35. $\dfrac{54b^2x^3y^4}{27b^3x^3y^5}$.

36. $\dfrac{5x^2y^3z^2u}{7ax^2y^4z^3}$.

37. $\dfrac{4a^3bc^2}{3a^2bc^2}$.

38. $\dfrac{29b^2x^2y^3}{58b^3x^2y^4}$.

39. $\dfrac{4a^2b^4x^3y}{6b^3x^3yz^2}$.

40. $\dfrac{x-1}{x^2-1}$.

41. $\dfrac{a^2-1}{a^2-2a+1}$.

42. $\dfrac{x^2-1}{x^3-1}$.

43. $\dfrac{1-a^2}{a^2-2a+1}$.

44. $\dfrac{x^2+1}{x^4-1}$.

45. $\dfrac{a^2-1}{a^2+2a+1}$.

46. $\dfrac{x^2-1}{x^3+1}$.

47. $\dfrac{x^2-1}{x^2-3x+2}$.

48. $\dfrac{x^2-1}{x^6-1}$.

49. $\dfrac{a^2-4}{a^2+3a+2}$.

50. $\dfrac{x^2-y^2}{x^3-y^3}$.

51. $\dfrac{a^3-a}{a^2-2a+1}$.

52. $\dfrac{x^2+y^2}{x^4-y^4}$.

53. $\dfrac{3x^2-3}{x^2+2x+1}$.

54. $\dfrac{x^2-y^2}{x^3+y^3}$.

55. $\dfrac{3x^2+6x}{x^2+4x+4}$.

56. $\dfrac{x^2-y^2}{x^6-y^6}$.

57. $\dfrac{a^2-b^2}{a^2-2ab+b^2}$.

58. $\dfrac{x^2-y^2}{(x+y)^2}$.

59. $\dfrac{a^2-x^2}{a^2+2ax+x^2}$.

60. $\dfrac{x^3+y^3}{(x+y)^3}$.

61. $\dfrac{a^3-ax^2}{a^2-2ax+x^2}$.

62. $\dfrac{(x+y)^3}{(x^2-y^2)^2}$.

63. $\dfrac{x^3-y^3}{x^2-2xy+y^2}$.

64. $\dfrac{(x+y)^2}{(x^2-y^2)^2}$.

65. $\dfrac{x^3+y^3}{x^2+2xy+y^2}$.

66. $\dfrac{(x-y)^3}{(x^2-y^2)^2}$.

67. $\dfrac{x^3-a^2x}{a^2+2ax+x^2}$.

68. $\dfrac{x^3-1}{x^4-1}$.

69. $\dfrac{x^2+x-6}{x^2+6x+9}$.

70. $\dfrac{x^4-1}{x^6+1}$.

71. $\dfrac{x^2-4x+4}{x^2-5x+6}$.

72. $\dfrac{x^6-1}{x^8-1}$.

73. $\dfrac{x^2+2x+1}{x^2-3x-4}$.

74. $\dfrac{x^6+1}{x^9+1}$.

75. $\dfrac{a^2-16}{a^2-2a-8}$.

76. $\dfrac{a^4-x^4}{a^6-x^6}$.

77. $\dfrac{x^2+3x+2}{x^2+6x+5}$.

78. $\dfrac{a^6+x^6}{a^8-x^8}$.

79. $\dfrac{x^2+2x-3}{x^2+4x+3}$.

80. $\dfrac{a^3-x^3}{a^9-x^9}$.

81. $\dfrac{a^2+2a-3}{a^2+5a+6}$.

82. $\dfrac{2x^2}{6x^2-2x}$.

83. $\dfrac{x^2-5x+6}{x^2+x-12}$.

84. $\dfrac{2a+2}{3a^2+3a}$.

85. $\dfrac{x^2-5x}{x^2-4x-5}$.

86. $\dfrac{a^2-a}{5a^2-5a}$.

87. $\dfrac{x^3+27}{x^2-2x-15}$.

88. $\dfrac{2x+3}{8x^3+27}$.

89. $\dfrac{3x^3y-9x^2y}{x^2-8x+15}$.

90. $\dfrac{x^2-4y^2}{(x-2y)^2}$.

91. $\dfrac{ax^4+64ax}{x^2-4x+16}$.

92. $\dfrac{a^2-x^2}{3a^2+3ax}$.

93. $\dfrac{x^2+x-90}{x^2+8x-20}$.

94. $\dfrac{4a^2-9x^2}{4a^2+6ax}$.

95. $\dfrac{a^2+5a+6}{a^2-4a-12}$.

96. $\dfrac{a^2+a}{2b+2ab}$.

97. $\dfrac{a^2+a-12}{2a^2+3a-20}$.

98. $\dfrac{a^2-1}{2ax+2x}$.

99. $\dfrac{a^2-a-20}{2a^2-7a-15}$.

100. $\dfrac{3x^2-6x}{2xy-4y}$.

101. $\dfrac{6x^2-5x-6}{8x^2-2x-15}$.

102. $\dfrac{3a^4x^4+3}{4a^8x^8-4}$.

103. $\dfrac{x^2+2x-24}{x^2-12x+32}$.

104. $\dfrac{6a^2-4ab}{9a^2b-6b^2}$.

105. $\dfrac{a^2-3a-28}{a^2-11a+28}$.

106. $\dfrac{2x^3-4x^2y}{4xy^2-8y^3}$.

107. $\dfrac{x^2+x-72}{x^2-10x+16}$.

108. $\dfrac{6a^3+3a^2b}{3b^3+6ab^2}$.

109. $\dfrac{x^2-2x-15}{x^2+10x+21}$.

110. $\dfrac{2a^3+2a^2b}{6b^3+6ab^2}$.

111. $\dfrac{2a^2+a-15}{2a^2-19a+35}$.

112. $\dfrac{2x^3+6x^2y}{9y^4+3xy^3}$.

113. $\dfrac{3a^2+23a-36}{4a^2+33a-27}$.

114. $\dfrac{a^2-x^2}{7a^3+7a^2x}$.

115. $\dfrac{6x^2+x-15}{6x^2-11x-35}$.

116. $\dfrac{6xy^2+6y^3}{3y^4+3x^3y}$.

117. $\dfrac{6a^2-11a-10}{6a^2-19a+10}$.

118. $\dfrac{6x^4+12x^3}{16x^2+2x^4}$.

119. $\dfrac{a^2-8a+15}{2a^2-13a+21}$.

120. $\dfrac{4x^4y-4x^6y}{2x^5y-2x^2y}$.

121. $\dfrac{2x^2+17x+21}{3x^2+26x+35}$.

122. $\dfrac{x^4y+x^6y}{3x^2y+3x^5y}$.

123. $\dfrac{8x^3+27y^3}{9y^2+12xy+4x^2}$.

To the Teacher. — Teach 175.

Reduce these fractions to entire or mixed quantities:

1. $\dfrac{a^3}{a+x}$.	2. $\dfrac{x^4+1}{x+1}$.	3. $\dfrac{a^3+x^3}{a+x}$.
4. $\dfrac{3x^2}{x-y}$.	5. $\dfrac{a^5-1}{a-1}$.	6. $\dfrac{x^3+y^3}{x-y}$.
7. $\dfrac{a+x}{a-x}$.	8. $\dfrac{x^5-1}{x+1}$.	9. $\dfrac{a^3-x^3}{a-x}$.
10. $\dfrac{a-x}{a+x}$.	11. $\dfrac{x^5+1}{x-1}$.	12. $\dfrac{x^3-y^3}{x+y}$.
13. $\dfrac{x^3+1}{x}$.	14. $\dfrac{a^5+1}{a+1}$.	15. $\dfrac{x^4-y^4}{x-y}$.
16. $\dfrac{a^4-1}{a^2}$.	17. $\dfrac{x^2-4}{x-2}$.	18. $\dfrac{a^4-x^4}{a+x}$.
19. $\dfrac{x^2-1}{x-1}$.	20. $\dfrac{a^2-4}{a+2}$.	21. $\dfrac{x^4+y^4}{x-y}$.
22. $\dfrac{a^3-1}{a+1}$.	23. $\dfrac{x^2+4}{x-2}$.	24. $\dfrac{a^4+x^4}{a+x}$.
25. $\dfrac{x^3-1}{x-1}$.	26. $\dfrac{x^2+4}{x+2}$.	27. $\dfrac{a^5-x^5}{a-x}$.
28. $\dfrac{a^3+1}{a+1}$.	29. $\dfrac{x^3-8}{x-2}$.	30. $\dfrac{a^5-x^5}{a+x}$.
31. $\dfrac{x^3+1}{x-1}$.	32. $\dfrac{a^3-8}{a+2}$.	33. $\dfrac{x^5+y^5}{x-y}$.
34. $\dfrac{a^4-1}{a-1}$.	35. $\dfrac{x^3+8}{x-2}$.	36. $\dfrac{a^5+x^5}{a+x}$.
37. $\dfrac{x^4+1}{x-1}$.	38. $\dfrac{a^3+8}{a+2}$.	39. $\dfrac{x^9+y^3}{x^3-y}$.

40. $\frac{15a^2+5a-1}{5a}$.

41. $\frac{a^2+3a-10}{a+2}$.

42. $\frac{12x^2-4x+5}{2x}$.

43. $\frac{x^2-7x+12}{x-3}$.

44. $\frac{9a^2+15a-4}{3a}$.

45. $\frac{x^2-5x-11}{x+5}$.

46. $\frac{6x^3-18x-2}{6x}$.

47. $\frac{a^2-9a-10}{a+1}$.

48. $\frac{16a^3+8a-3}{4a}$.

49. $\frac{x^2-5x-12}{x-3}$.

50. $\frac{4x^4-16x+7}{2x}$.

51. $\frac{a^2+8a+15}{a+5}$.

52. $\frac{18a^2+6a-3}{3a}$.

53. $\frac{x^2+9x-20}{x+4}$.

54. $\frac{24x^3-6y-5}{6x}$.

55. $\frac{a^2-6a-27}{a+3}$.

56. $\frac{4a^3+20a+2}{4a}$.

57. $\frac{4x^2-3x+2}{x+2}$.

58. $\frac{5x^3-15y-4}{5x}$.

59. $\frac{2a^2-9a+9}{a-3}$,

60. $\frac{35a^2+7a-3}{7a}$.

61. $\frac{5x^2-9x-8}{x-4}$.

62. $\frac{24x^3-6y-2}{3x}$.

63. $\frac{4a^2+7a-2}{a+2}$.

64. $\frac{2a^2+18a+3}{2a}$.

65. $\frac{8x^2+2x-9}{4x-3}$.

66. $\frac{30x^3-5y-4}{5x}$.

67. $\frac{6a^2+9a+6}{2a+5}$.

TO THE TEACHER. — Teach 176.

Reduce these mixed quantities to the fractional form:

1. $x+1+\frac{4x+3}{3x}$.

2. $\frac{4a-3x}{5}-3a+2x$.

3. $a-2-\frac{2a-3}{2a}$.

4. $\frac{2x+5y}{4}-2x-3y$.

5. $x-3+\frac{5x+2}{4x}$.

6. $3a-5b-\frac{6a-4b}{3}$.

7. $a-5-\frac{3a+4}{a-5}$.

8. $\frac{7x+4y}{2}-3x-4y$.

9. $x+2+\frac{4x-5}{x+2}$.

10. $\frac{2a-6x}{5}-3a+2x$.

11. $a-4-\frac{a-16}{a+4}$.

12. $5x+4y-\frac{8x+7y}{4}$.

13. $x+5-\frac{x-25}{x-5}$.

14. $\frac{9a-8b}{6}-3a-2b$.

15. $a+x-\frac{a^2+x^2}{a+x}$.

16. $\frac{8x^2+7y^2}{3x-2y}-2x-3y$.

17. $x-y+\frac{x^2+y^2}{x+y}$.

18. $2a-4x-\frac{6a^2-9x^2}{4a+3x}$.

19. $a-x-\frac{a^2+x^2}{a-x}$.

20. $\frac{6x^2-8y^2}{5x-3y}-3x-4y$.

21. $x+y+\frac{x^2+y^2}{x-y}$.

22. $\frac{4a^2+9x^2}{3a-2x}-4a+3x$.

23. $a+c-\frac{2ac+c^2}{a+c}$.

24. $3x-2y-\frac{8x^2-7y^2}{2x-3y}$.

25. $c+d^2-\frac{ac+bd^2}{a+b}$.

26. $\frac{8a^2-7b^2}{4a+6b}-3a-4b$.

To the Teacher. — Teach 177.

Reduce these fractions to a common denominator:

1. $\frac{2a^2}{3}, \frac{3ax}{2}, \frac{5xy}{6}.$

2. $\frac{a+1}{a}, \frac{a-1}{b}, \frac{a^2-1}{c}.$

3. $\frac{5ac}{b}, \frac{3x^2}{a}, \frac{4ax}{c}.$

4. $\frac{3}{x-1}, \frac{2}{x+1}, \frac{5}{x^2-1}.$

5. $\frac{4}{3ab}, \frac{3}{4a^2}, \frac{b}{6ax}.$

6. $\frac{x+2}{a}, \frac{x-5}{3}, \frac{x^2+4}{b}.$

7. $\frac{5a^2}{3}, \frac{4ab}{c}, \frac{2xy}{a}.$

8. $\frac{a}{x-3}, \frac{b}{x+3}, \frac{c}{x^2-9}.$

9. $\frac{a}{3bx}, \frac{4}{3a^2}, \frac{5ab}{2}.$

10. $\frac{4}{2-x}, \frac{5}{2+x}, \frac{3}{4-x^2}.$

11. $\frac{4ax}{3}, \frac{a}{3x^2}, \frac{4bx}{c}.$

12. $\frac{a-1}{a}, \frac{b-2}{b}, \frac{x^2-4}{c}.$

13. $\frac{5}{3ax}, \frac{a}{2b^2}, \frac{c}{6ab}.$

14. $\frac{x-2}{x+2}, \frac{x+2}{x-2}, \frac{x^2+4}{x^2-4}.$

15. $\frac{2x^2}{b}, \frac{3bx}{2}, \frac{4}{3bc}.$

16. $\frac{5}{a-x}, \frac{3ac}{a^2-x^2}, \frac{7}{a+x}.$

17. $\frac{3}{5ax}, \frac{b}{2x^2}, \frac{3bx}{5}.$

18. $\frac{x-y}{x+y}, \frac{x+y}{x-y}, \frac{x^2+y^2}{x^2-y^2}.$

19. $\frac{3xy}{2}, \frac{3}{2x^2}, \frac{a}{4xy}.$

20. $\frac{a+b}{3}, \frac{a^2-x^2}{2a}, \frac{a-4}{6}.$

21. $\frac{x}{3y^2}, \frac{4}{3xy}, \frac{b}{2ax}.$

22. $\frac{a}{x+2}, \frac{b}{2x+4}, \frac{c}{4x+8}.$

23. $\frac{5ax}{b}, \frac{3b^2}{a}, \frac{5}{3ax}.$

24. $\frac{x+2}{x^2-3x-4}, \frac{x-1}{x^2-2x-8}.$

25. $\frac{x}{3ab}, \frac{y}{4x^2}, \frac{z}{2bc}.$

26. $\frac{x+3}{x^2-5x+6}, \frac{x-2}{x^2-6x+9}.$

TO THE TEACHER.—Teach 178 . . . . 184.

ADDITION AND SUBTRACTION OF FRACTIONS.

1. $\frac{a+x}{4}+\frac{a-x}{5}$.

2. $\frac{a+b}{2}+\frac{b+c}{3}+\frac{a+c}{2}$.

3. $\frac{a+b}{a}+\frac{b-x}{b}$.

4. $\frac{x+y}{4}+\frac{x-y}{2}-\frac{x+y}{8}$.

5. $\frac{a-x}{3}-\frac{a+x}{5}$.

6. $\frac{a+x}{3}+\frac{a-x}{6}+\frac{x-a}{2}$.

7. $\frac{1}{1+x}+\frac{1}{1-x}$.

8. $\frac{x-y}{2}-\frac{x-y}{3}-\frac{x+y}{4}$.

9. $\frac{1}{x+3}-\frac{1}{x+8}$.

10. $\frac{5}{a+x}+\frac{3}{a-x}+\frac{4}{a+x}$.

11. $\frac{a+4}{3}+\frac{5-a}{5}$.

12. $\frac{x}{x-y}+\frac{y}{x+y}+\frac{x-y}{x+y}$.

13. $\frac{x+3}{3}-\frac{4-x}{6}$.

14. $\frac{x+8}{2}-\frac{x+9}{3}-\frac{x+4}{4}$.

15. $\frac{a}{a+b}+\frac{b}{a-b}$.

16. $\frac{5}{a+x}+\frac{8}{a-x}-\frac{3x}{a^2-x^2}$.

17. $\frac{x+y}{4}-\frac{x+4}{5}$.

18. $\frac{x+y}{x-y}+\frac{x}{x+y}-\frac{2x^2}{x^2-y^2}$.

19. $\frac{1}{a-b}+\frac{1}{a+b}$.

20. $\frac{a+x}{a}+\frac{a-x}{x}+\frac{a^2-x^2}{ax}$.

21. $\frac{a+x}{a-x}-\frac{a-x}{a+x}$.

22. $\frac{x}{x-y}-\frac{y}{x+y}-\frac{y^2}{x^2-y^2}$.

23. $\frac{x-2}{x+2}-\frac{x+2}{x-2}$.

24. $\frac{4}{x+4}+\frac{4}{x-4}-\frac{8x}{x^2-16}$.

25. $\frac{a+b}{a-b}+\frac{a}{a+b}$.

26. $\frac{x+5}{x-5}-\frac{x-5}{x+5}-\frac{19x+5}{x^2-25}$.

TO THE TEACHER.—Teach 185.

1. $\dfrac{4a^4}{a^2+x}+\dfrac{4a^2x}{a^2-x}.$

2. $\dfrac{a+x}{4}-\dfrac{a-x}{4}-\dfrac{2x^2}{a+x}.$

3. $\dfrac{1+x^2}{1-x^2}-\dfrac{1-x^2}{1+x^2}.$

4. $\dfrac{2}{x-1}-\dfrac{3}{x+1}-\dfrac{x^2-3}{1-x^2}.$

5. $\dfrac{x^2+1}{x^2-1}+\dfrac{4x^2}{1-x^4}.$

6. $\dfrac{5}{x-2}-\dfrac{4}{2+x}+\dfrac{16}{4-x^2}.$

7. $\dfrac{1}{x^2-3}-\dfrac{1}{x^2+2}.$

8. $\dfrac{x+3}{3-x}-\dfrac{x-3}{x+3}+\dfrac{8x}{x^2-9}.$

9. $\dfrac{3}{x^2-2}+\dfrac{3x^2}{4-x^4}.$

10. $\dfrac{5}{a+x}-\dfrac{8}{a-x}+\dfrac{3x}{x^2-a^2}.$

11. $\dfrac{x^2+4}{x^2+3}-\dfrac{x^2+3}{x^2+2}.$

12. $\dfrac{a+x}{a-x}-\dfrac{a}{a+x}+\dfrac{x^2}{x^2-a^2}.$

13. $\dfrac{2}{x^2+3x}+\dfrac{6x}{x^2-9}.$

14. $\dfrac{x}{x-y}-\dfrac{y}{x+y}-\dfrac{y^2}{y^2-x^2}.$

15. $\dfrac{1}{2x-x^2}-\dfrac{1}{x^2-4}.$

16. $\dfrac{3}{x+4}-\dfrac{3}{x-4}-\dfrac{24}{16-x^2}.$

17. $\dfrac{4x-1}{2x+2}+\dfrac{6x+2}{3+3x}.$

18. $\dfrac{x-5}{x+5}+\dfrac{x+5}{x-5}+\dfrac{x^2+45}{25-x^2}.$

19. $\dfrac{3x+1}{3x-3}+\dfrac{2x-3}{4-4x}.$

20. $\dfrac{x^2+x+1}{x+1}+\dfrac{x^2-x+1}{x-1}.$

21. $\dfrac{x}{6+3x}+\dfrac{2}{2x+4}.$

22. $\dfrac{x-1}{x^2-x+1}-\dfrac{1+x}{x^2+x+1}.$

23. $\dfrac{4x+3}{5-5x}-\dfrac{2x-1}{3x-3}.$

24. $\dfrac{1+x}{1+x+x^2}-\dfrac{1-x}{1-x+x^2}.$

25. $\dfrac{1}{a^2+ax}-\dfrac{1}{x^2+ax}.$

26. $\dfrac{a+3}{a^2+2a-8}-\dfrac{a-3}{a^2+a-6}.$

27. $\dfrac{2}{a^2-ab}+\dfrac{3}{b^2-ab}.$

28. $\dfrac{5}{1-2x}-\dfrac{4}{2x+1}+\dfrac{16x}{4x^2-1}.$

29. $\dfrac{a+c}{c}-\dfrac{a}{a+c}-\dfrac{a(a^2-c)}{c(a^2-c^2)}.$

30. $\dfrac{y}{x+y}-\dfrac{y^2}{(x+y)^2}-\dfrac{x^2y}{(x+y)^3}.$

31. $\dfrac{4}{x-y}+\dfrac{4y}{(x-y)^2}+\dfrac{4xy}{(x-y)^3}.$

32. $\dfrac{2}{(a+x)^2}+\dfrac{2}{(a-x)^2}+\dfrac{4}{x^2-a^2}.$

33. $\dfrac{2}{x(x+2)}+\dfrac{3}{x(x-2)}-\dfrac{4}{x^2-4}.$

34. $\dfrac{2}{x-y}-\dfrac{x+y}{x^2+xy+y^2}+\dfrac{4xy}{y^3-x^3}.$

35. $\dfrac{x-2}{x+3}+\dfrac{x+4}{x-5}-\dfrac{x^2+31}{x^2-2x-15}.$

36. $\dfrac{a-x}{a^2-ax+x^2}-\dfrac{1}{a+x}+\dfrac{3a^2}{x^3+a^3}.$

37. $\dfrac{a-x}{a-b}+\dfrac{a-b}{a-x}-\dfrac{(a-b)^2}{(a-x)(a-b)}.$

38. $\dfrac{x+4}{x^2+x+1}-\dfrac{1}{x-1}-\dfrac{x^2+4x-2}{1-x^3}.$

39. $\dfrac{x^2+3x+5}{x^3+1}+\dfrac{2}{x+1}-\dfrac{3x+6}{x^2-x+1}.$

40. $\dfrac{x+4}{x^2-3x+9}+\dfrac{x^2-4x+9}{x^3+27}-\dfrac{2}{x+3}.$

41. $\dfrac{3}{x-2}-\dfrac{x+6}{x^2+2x+4}+\dfrac{x^2+2x+28}{8-x^3}.$

42. $\dfrac{1}{a+2x}+\dfrac{a-2x}{a^2-2ax+4x^2}+\dfrac{4ax}{a^3+8x^3}.$

43. $\frac{3(x+1)}{x^2-8x-20}-\frac{x-4}{x^2-18x+80}-\frac{x+5}{x^2-6x-16}.$

44. $\frac{2(x-3)}{x^2+6x-16}-\frac{x-2}{x^2+4x-12}+\frac{2}{x^2+14x+48}.$

45. $\frac{a^2+ac+c^2}{(a-b)(b-c)}-\frac{a^2+ab+b^2}{(a-c)(c-b)}-\frac{b^2+bc+c^2}{(a-c)(b-a)}.$

46. $\frac{1}{(a-b)(a-c)}+\frac{1}{(a-b)(b-c)}+\frac{1}{(a-c)(b-c)}.$

47. $\frac{3}{(a-b)(b-c)}-\frac{4}{(a-c)(a-b)}-\frac{3}{(c-a)(c-b)}.$

48. $\frac{a+c}{(a-3)(3-c)}-\frac{a+3}{(a-c)(c-3)}-\frac{c+3}{(a-c)(3-a)}.$

49. $\frac{a+1}{(a-b)(a-c)}+\frac{b+1}{(b-a)(b-c)}-\frac{c+1}{(c-a)(b-c)}.$

50. $\frac{a}{(a-b)(a-c)}+\frac{b}{(b-c)(b-a)}+\frac{c}{(c-a)(c-b)}.$

51. $\frac{a^2}{(a-b)(a-c)}-\frac{b^2}{(b-a)(c-b)}-\frac{c^2}{(c-a)(b-c)}.$

52. $\frac{c}{(a-b)(a-c)}-\frac{c}{(c-b)(b-a)}-\frac{c}{(b-c)(c-a)}.$

53. $\frac{a+c}{(a-b)(b-c)}-\frac{b+c}{(c-a)(b-a)}-\frac{a+b}{(a-c)(c-b)}.$

54. $\frac{x^2-yz}{(x-y)(x-z)}+\frac{y^2+xz}{(y-x)(y+z)}+\frac{z^2+xy}{(z-x)(z+y)}.$

55. $\frac{x-2}{(x-1)(x-3)}+\frac{x-1}{(x-2)(3-x)}+\frac{x-3}{(1-x)(2-x)}.$

56. $\frac{1}{a(a-c)(a-x)}+\frac{1}{c(c-a)(c-x)}+\frac{1}{x(x-c)(x-a)}.$

TO THE TEACHER. — Teach 186 and 187.

MULTIPLICATION OF FRACTIONS.

1. $\dfrac{4ab}{3cd} \times \dfrac{2ac}{5bd} \times \dfrac{5bd}{8ac}$.

2. $\dfrac{x-1}{a} \times \dfrac{x+1}{b}$.

3. $\dfrac{6ax}{5by} \times \dfrac{4cy}{3ad} \times \dfrac{5bd}{2cx}$.

4. $\dfrac{x+1}{4} \times \dfrac{8a}{x^2-1}$.

5. $\dfrac{9ac}{8bd} \times \dfrac{2by}{3cx} \times \dfrac{4bx}{3ay}$.

6. $\dfrac{1-x}{b} \times \dfrac{ac}{1-a^2}$.

7. $\dfrac{2bc}{3ax} \times \dfrac{9ax}{4bc} \times \dfrac{2xy}{5ab}$.

8. $\dfrac{1+x}{a-1} \times \dfrac{a-1}{1-x^2}$.

9. $\dfrac{3ab}{4xy} \times \dfrac{2bc}{5ax} \times \dfrac{6ad}{7bx}$.

10. $\dfrac{a+2}{a-1} \times \dfrac{a^2-1}{a^2-4}$.

11. $\dfrac{4xy}{3ab} \times \dfrac{9a^2}{8cy} \times \dfrac{2bc}{3ax}$.

12. $\dfrac{a+x}{a-c} \times \dfrac{a^2-c^2}{a^2+ax}$.

13. $\dfrac{4ab}{7x^2} \times \dfrac{5cy}{8bd} \times \dfrac{6x^3}{7ay}$.

14. $\dfrac{x+y}{x-y} \times \dfrac{x^2-y^2}{(x+y)^2}$.

15. $\dfrac{2c^2}{5ab} \times \dfrac{3bd}{4ac} \times \dfrac{5a^2}{6cd}$.

16. $\dfrac{x^2-9}{x^2+4x} \times \dfrac{x^2-16}{x^2-3x}$.

17. $\dfrac{5x^4}{3ad} \times \dfrac{9ab}{5x^2} \times \dfrac{7cy}{6bx}$.

18. $\dfrac{3x^2+x}{x-2} \times \dfrac{x-1}{2x+6x^2}$.

19. $\dfrac{2a^3}{7bx} \times \dfrac{3bx}{8ay} \times \dfrac{5cd}{6ax}$.

20. $\dfrac{a-x}{a^2+2ax} \times \dfrac{a^2-4x^2}{a^2-ax}$.

21. $\dfrac{4x^4}{9bc} \times \dfrac{2ac}{3x^2} \times \dfrac{6b^3}{5ax}$.

22. $\dfrac{a^2-b^2}{a^2-9x^2} \times \dfrac{3ax+a^2}{b^2+ab}$.

23. $\dfrac{8bx}{9a^2} \times \dfrac{3a^4}{2xy} \times \dfrac{3cy}{4a^2}$.

24. $\dfrac{(a+b)^2}{a+2b} \times \dfrac{a^3+8b^3}{(a^2-b^2)^2}$.

25. $\dfrac{2x^5}{3by} \times \dfrac{5ab}{7x^2} \times \dfrac{4y^3}{5ax}$.

26. $\dfrac{(a^2-16)}{xy^3+8x^4} \times \dfrac{2x+y}{(a+4)^2}$.

27. $\left(x-\frac{x^2}{a}\right)\times\left(\frac{1}{x}-\frac{1}{a}\right).$

28. $\left(1+\frac{x}{1+x}\right)\left(1+\frac{3x}{1-x}\right).$

29. $\left(a-\frac{x^2}{a}\right)\times\left(\frac{a}{x}+\frac{x}{a}\right).$

30. $\left(1-\frac{2a}{1+a}\right)\left(1+\frac{2a}{1-a}\right).$

31. $\left(b+\frac{a^2}{b}\right)\times\left(a-\frac{b^2}{a}\right).$

32. $\left(1-\frac{a-b}{a+b}\right)\left(2+\frac{2b}{a-b}\right).$

33. $\left(b+\frac{bx}{a}\right)\left(1-\frac{a}{a+x}\right).$

34. $\left(a+\frac{ab}{a-b}\right)\left(b-\frac{ab}{a+b}\right).$

35. $\left(4+\frac{2a}{3x}\right)\times\left(2-\frac{2a}{6x}\right).$

36. $\left(1+\frac{a+x}{a-x}\right)\left(1-\frac{a-x}{a+x}\right).$

37. $\left(\frac{a}{b}+\frac{b}{2a}\right)\times\frac{4ab}{b^2+2a^2}.$

38. $\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\frac{a^2bc}{ac+ab+bc}.$

39. $\frac{3ax}{4by}\times\frac{a^2-x^2}{c^2-x^2}\times\frac{bx+bc}{ax+a^2}\times\frac{c-x}{a-x}.$

40. $\frac{3ax}{4by}\times\frac{2}{(a+x)^2}\times\frac{(a^2-x^2)^2}{6a^2}\times\frac{8b}{a-x}.$

41. $\frac{x-y}{(x+y)^2}\times\frac{x+y}{(x-y)^2}\times\frac{2ax}{y}\times\frac{xy-y^2}{x^2+xy}.$

42. $\frac{(x-y)^2}{2+x}\times\frac{x-3}{x^2-y^2}\times\frac{(y+x)^2}{x^3-27}\times\frac{x^3+8}{x^2-y^2}.$

43. $\frac{a^2-b^2}{ab^2x}\times\frac{ab-b^2}{(a+b)^2}\times\frac{ab+a^2}{(a-b)^2}\times\frac{x^2+4x}{a+x}.$

44. $\frac{x^2-6x+8}{x^2-4x+3}\times\frac{x^2-5x+6}{x^2-2x-8}\times\frac{2x+x^2}{x^2-4x+4}.$

45. $\frac{x^4-8x}{x^2-4x-5}\times\frac{x^2+2x+1}{x^3-x^2-2x}\times\frac{x-5}{x^2+2x+4}.$

46. $\frac{a^2-ax+x^2}{2a^3-3a^2x}\times\frac{4a^2-9x^2}{a^3+x^3}\times\frac{a^2+ax}{6x+4a}\times\frac{3ab}{2xy}.$

TO THE TEACHER.—Teach 188 . . . . 190.

DIVISION OF FRACTIONS.

1. $\dfrac{8a^2x}{5b^3y} \div \dfrac{4ax}{5by}.$

2. $\dfrac{a-1}{a} \div \dfrac{b}{a+1}.$

3. $\dfrac{9a^3}{8b^4x} \div \dfrac{3a^2}{2bx}.$

4. $\dfrac{a+1}{4} \div \dfrac{a^2-1}{8a}.$

5. $\dfrac{6x^2}{7by^2} \div \dfrac{3ax}{4by}.$

6. $\dfrac{1-a}{b} \div \dfrac{1-a^2}{ac}.$

7. $\dfrac{4a^2x}{3bc^2} \div \dfrac{8ax}{9bc}.$

8. $\dfrac{1+a}{x-1} \div \dfrac{1-a^2}{x-1}.$

9. $\dfrac{8ac^3}{9b^2x} \div \dfrac{4ac}{9bx}.$

10. $\dfrac{x+2}{x+1} \div \dfrac{x^2-4}{x^2-1}.$

11. $\dfrac{7a^3b}{12c^2} \div \dfrac{7ab}{4cx}.$

12. $\dfrac{a+x}{a-c} \div \dfrac{a^2+ax}{a^2-c^2}.$

13. $\dfrac{14x^2y}{15ab^2} \div \dfrac{7xy}{5ab}.$

14. $\dfrac{a+b}{a-b} \div \dfrac{(a+b)^2}{a^2-b^2}.$

15. $\dfrac{9ab}{28x^2y} \div \dfrac{3ac}{7xy}.$

16. $\dfrac{a^2-9}{a^2+4a} \div \dfrac{a^2-3a}{a^2-16}.$

17. $\dfrac{25xy^3}{27a^2b} \div \dfrac{5xy^2}{9ab}.$

18. $\dfrac{3x^2+x}{x-2} \div \dfrac{2x+6x^2}{x-1}.$

19. $\dfrac{35a^2x^3}{24b^2y^3} \div \dfrac{7a^2x}{8b^2y}.$

20. $\dfrac{a-x}{a^2+2ax} \div \dfrac{a^2-ax}{a^2-4x^2}.$

21. $\dfrac{8a^2}{a^2-x^2} \div \dfrac{4a}{a+x}.$

22. $\dfrac{x^2-y^2}{x^2-9c^2} \div \dfrac{y^2+xy}{3cx+x^2}.$

23. $\dfrac{12x^2y}{5x-10} \div \dfrac{6xy}{x^2-2x}.$

24. $\dfrac{(a+b)^2}{a+2b} \div \dfrac{(a^2-b^2)^2}{a^3+8b^3}.$

25. $\dfrac{15a^3b^2}{4x+8y} \div \dfrac{5a^2b^2}{x^2+2xy}.$

26. $\dfrac{(a^2-16)^2}{xy^3+8x^4} \div \dfrac{(a+4)^2}{2x+y}.$

27. $\left(\frac{a}{b}+\frac{1}{b}\right)\div\left(a-\frac{1}{a}\right).$

28. $\frac{x^2-2x+4}{x-3}\div\frac{x^3+8}{x^2-9}.$

29. $\left(\frac{a}{b}-\frac{b}{a}\right)\div\left(1+\frac{b}{a}\right).$

30. $\frac{a^2-x^2}{x^2-5x-6}\div\frac{a+x}{x^2+x}.$

31. $\left(\frac{1}{b}+\frac{1}{a}\right)\div\left(1+\frac{a}{b}\right).$

32. $\frac{a^2-a-6}{a^2-a-2}\div\frac{2a+a^2}{a^2-2a}.$

33. $\left(1+\frac{2}{a}\right)\div\left(1+\frac{a}{2}\right).$

34. $\frac{a^3-b^3}{x^2+6x+8}\div\frac{a-b}{8+2x}.$

35. $\left(a-\frac{b^2}{a}\right)\div\left(\frac{a}{b}+1\right).$

36. $\frac{a^3+b^3}{a^3-b^3}\div\frac{a+b}{a^2+ab+b^2}.$

37. $\left(1-\frac{b^2}{a^2}\right)\div\left(1+\frac{a}{b}\right).$

38. $\frac{a^2+2ab+b^2}{ab-b^2}\div\frac{a^2-b^2}{b^2}.$

39. $\left(a-\frac{c^2}{a}\right)\div\left(a+\frac{b}{a}\right).$

40. $\frac{x^2-x-2}{x^2-9}\div\frac{x^2+x-6}{x-3}.$

41. $\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\div\left(\frac{b}{a}+1\right).$

42. $\frac{a^3+x^3}{a^2-2ax+x^2}\div\frac{ax+x^2}{a-x}.$

43. $\left(a^2+\frac{b^3}{a}\right)\div\left(1+\frac{a}{b}\right).$

44. $\frac{x^2+x-6}{x^2-16}\div\frac{x^2-x-12}{4x+x^2}.$

45. $\left(a^2-\frac{b^3}{a}\right)\div\left(1-\frac{b}{a}\right).$

46. $\frac{x^2+7x-8}{2a+x}\div\frac{x^2+x-56}{ax^2+2a^2x}.$

47. $\left(\frac{a}{b^2}+\frac{b}{a^2}\right)\div\left(\frac{b}{a}+1\right).$

48. $\left(4-\frac{2}{a+1}\right)\div\left(8+\frac{6}{a^2-1}\right).$

49. $\left(1+\frac{b^2}{a^2}\right)\div\left(\frac{a^4}{b^4}-1\right).$

50. $\left(a+\frac{3a}{a^2-4}\right)\div\left(a+\frac{a}{a-2}\right).$

51. $\left(\frac{1}{b^4}-\frac{1}{a^4}\right)\div\frac{c(a^2-b^2)}{a^3b^3}.$

52. $\left(3+\frac{5x}{a-x}\right)\div\left(9+\frac{5x^2}{a^2-x^2}\right).$

53. $\left(a^3-\frac{b^4}{a}\right)\div\frac{c(a^2+b^2)}{a}.$

54. $\left(4+\frac{15}{x^2-2^2}\right)\div\left(2-\frac{3}{x+2}\right).$

To the Teacher.—Teach 191.

COMPLEX FRACTIONS.

1. $\dfrac{a+\dfrac{b}{x}}{a-\dfrac{b}{c}}$.

2. $\dfrac{x-3+\dfrac{2}{x}}{x+1-\dfrac{6}{x}}$.

3. $\dfrac{\dfrac{1}{1-a}+\dfrac{1}{1+x}}{\dfrac{1}{1-a}-\dfrac{1}{1+x}}$.

4. $\dfrac{x+\dfrac{y}{2}}{a-\dfrac{x}{3}}$.

5. $\dfrac{a+2+\dfrac{1}{a}}{1+\dfrac{1}{a}}$.

6. $\dfrac{x-2-\dfrac{14}{x+3}}{x-1-\dfrac{21}{x+3}}$.

7. $\dfrac{a-\dfrac{1}{a}}{1+\dfrac{1}{a}}$.

8. $\dfrac{1-\dfrac{1}{a+1}}{1+\dfrac{1}{a-1}}$.

9. $\dfrac{\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}$.

10. $\dfrac{\dfrac{1}{a}+\dfrac{1}{b}}{\dfrac{b}{a}-\dfrac{a}{b}}$.

11. $\dfrac{\dfrac{a}{b}-\dfrac{b}{c}-\dfrac{c}{a}}{\dfrac{1}{c}-\dfrac{1}{a}-\dfrac{1}{b}}$.

12. $1-\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{a}}}$.

13. $\dfrac{\dfrac{a}{x}+\dfrac{y}{b}}{\dfrac{a}{x}-\dfrac{b}{c}}$.

14. $\dfrac{1+\dfrac{b}{a-b}}{1-\dfrac{b}{a+b}}$.

15. $\dfrac{1}{b+\dfrac{1}{1+\dfrac{b+1}{3-b}}}$.

16. $\dfrac{1+\dfrac{1}{a}}{1-\dfrac{1}{b}}$.

17. $\dfrac{1}{x+\dfrac{1}{x+\dfrac{1}{x}}}$.

18. $\dfrac{1-\dfrac{2y-2z}{x+y-z}}{1+\dfrac{2z}{x-y-z}}$.

19. $\dfrac{1+\dfrac{a}{b}}{1-\dfrac{b^2}{a^2}}$.

20. $\dfrac{a+\dfrac{3a}{a-4}}{a-\dfrac{3a}{a-4}}$.

21. $\dfrac{\dfrac{a+b}{c+d}+\dfrac{a-b}{c-d}}{\dfrac{a+b}{c-d}+\dfrac{a-b}{c+d}}$.

REVIEW OF FRACTIONS.

1. $\dfrac{a^6-x^6}{4a^4x^4}\left(\dfrac{a^3}{a^3+x^3}-1+\dfrac{x^3}{a^3-x^3}\right).$

2. $\left(\dfrac{a}{b}-\dfrac{a+2b}{a+b}\right)\div\left(\dfrac{a}{b}-2+\dfrac{a}{a+b}\right).$

3. $\dfrac{a^3+x^3}{a^3-x^3}\times\dfrac{a^2+ax+x^2}{a^2-ax+x^2}\div\dfrac{(a+x)^2}{(a-x)^2}.$

4. $\left(\dfrac{9+2x^2}{9-x^2}-\dfrac{3-x}{3+x}+1\right)\times\left(\dfrac{x+3}{2x+3}\right).$

5. $\left(\dfrac{x}{x+2}+\dfrac{2-x}{x}\right)\div\left(\dfrac{x}{2+x}-\dfrac{2-x}{x}\right).$

6. $\left(\dfrac{2}{a-x}-\dfrac{2}{a+3x}\right)\dfrac{1}{x}-\dfrac{6-3x-a}{a^2+2ax-3x^2}.$

7. $\dfrac{(a^2-2+a)3}{a^2-a-2}-\dfrac{(a^2-2-a)3}{a^2-2+a}-\dfrac{8a}{a^2-4}.$

8. $\dfrac{a^2+x^2-ax}{a^2+9x^2+3ax}\times\dfrac{a(a^2-x^2)}{a^3+x^3}\times\dfrac{a^3-27x^3}{a(a-x)}.$

9. $\left(\dfrac{12}{a-3}+1-\dfrac{4}{a-1}\right)\left(\dfrac{4}{a+1}+1-\dfrac{12}{a+3}\right).$

10. $\dfrac{(x-y)y}{x^2+y^2+2xy}\times\dfrac{x(y+x)}{x^2+y^2-2xy}\times\dfrac{ac(x^2-y^2)}{bx^2y^2}.$

11. $\left(\dfrac{x-1}{x^2-x+1}+\dfrac{x^2-1}{x^3+1}\right)\div\left(\dfrac{x+1}{x^2-x+1}-\dfrac{x^2-1}{x^3+1}\right).$

12. $\dfrac{2}{x}-\dfrac{x+y-z}{xy}+\dfrac{2}{y}-\dfrac{x+z-y}{xz}+\dfrac{2}{z}-\dfrac{y+z-x}{yz}.$

13. $\dfrac{2(b+c)}{(a-b)(a-c)}+\dfrac{2(a+c)}{(b-c)(b-a)}+\dfrac{2(a+b)}{(c-a)(c-b)}.$

14. $\left(a-x-\dfrac{a^2+x^2}{a+x}\right)\times\left(a+x-\dfrac{a^2+x^2}{a-x}\right).$

15. $\left(\dfrac{a+x}{a-x}-\dfrac{a^3+x^3}{a^3-x^3}\right)\times\left(\dfrac{a+x}{a-x}+\dfrac{a^2+x^2}{a^2-x^2}\right).$

16. $\dfrac{ax^4-8\,ax}{x^2-5\,x-6}\times\dfrac{x^2+1+2\,x}{x^3-x^2-2\,x}\div\dfrac{x^2+4+2\,x}{x^2-6\,x}.$

17. $\dfrac{(a+c)^2-x^2}{cx-ac+c^2}\times\dfrac{ac-c^2+cx}{a^2-ax+ac}\div\dfrac{(a+x)^2-c^2}{ac+ax-a^2}.$

18. $\left(2+\dfrac{a}{a-3}\right)\times\dfrac{9-a^2}{4-a^2}\times\dfrac{a+2}{a^2+a-6}-\dfrac{2}{a+2}.$

19. $\dfrac{(a+c)^2-x^2}{a^2+ac-ax}\times\dfrac{a}{(a+x)^2-c^2}\times\dfrac{(a-c)^2-x^2}{ac-c^2-cx}.$

20. $\left(\dfrac{x^2-14\,x+33}{y^2+10\,y-24}\div\dfrac{x^2+3\,x-18}{y^2+4\,y-12}\right)\times\dfrac{6\,y+xy}{6\,x+xy}.$

21. $\left(\dfrac{a-4}{a-3}+\dfrac{3-a}{9-a^2}-\dfrac{a}{a+3}\right)\times\left(\dfrac{a}{3}-1\right)\div\dfrac{5-a}{9-a^2}.$

22. $\dfrac{a}{a^2-x^2}+\dfrac{a^3+x^3}{x^4-a^4}-\dfrac{x}{a^2+x^2}+\dfrac{ax}{(x+a)(x^2+a^2)}.$

23. $\dfrac{4}{(x-1)(x-3)}+\dfrac{2}{(x-2)(3-x)}-\dfrac{2}{(2-x)(1-x)}.$

24. $\dfrac{a^2+2\,a-3}{a^2-a+30}\times\dfrac{a^2+6\,a+5}{a^3-a^2}\div\left(\dfrac{a^2+4\,a+3}{a^2-4\,a-12}\times\dfrac{2+a}{a^3}\right).$

25. $\left(1+\dfrac{a-1}{a+1}\right)\div\left(1-\dfrac{a-1}{a+1}\right)+\left(1+\dfrac{a+1}{a-1}\right)\div\left(1-\dfrac{a+1}{a-1}\right).$

26. $\dfrac{\dfrac{a}{b}+\dfrac{b}{a}}{\dfrac{a}{b}-\dfrac{b}{a}}+\dfrac{\dfrac{1}{a}+\dfrac{1}{b}}{\dfrac{a}{b}-\dfrac{b}{a}}.$

27. $\dfrac{\dfrac{1}{x+2}-\dfrac{6}{(x+4)(x+3)}}{\dfrac{1}{x+4}-\dfrac{5}{(x+3)(x+2)}}.$

28. $\left(\dfrac{x^2+y^2}{2xy}-1\right)\dfrac{xy^2}{x^3+y^3}+\dfrac{4x^2y-4xy^2}{x^2+y^2-xy}.$

29. $\left(\dfrac{x+1}{x^2+1}-\dfrac{x^2+1}{x^3+1}\right)\div\left(\dfrac{x^2+1}{x^3+1}-\dfrac{x^3+1}{x^4+1}\right).$

30. $\left(\dfrac{a^2}{x}-\dfrac{x^2}{a}\right)\left(\dfrac{a}{x^2}+\dfrac{x}{a^2}\right)\div\left(\dfrac{a^2}{x}+\dfrac{x^2}{a}\right)\left(\dfrac{a}{x^2}-\dfrac{x}{a^2}\right).$

31. $\dfrac{x^2+y^2+2xy-z^2}{x^2-2yz-y^2-z^2}\times\dfrac{x^2-2xz-y^2+z^2}{y^2-2yz-x^2+z^2}.$

32. $\left(\dfrac{2+n}{3+2n}-\dfrac{4n+5}{6+5n}\right)\div\left(\dfrac{3+2n}{4+3n}-\dfrac{4+3n}{4n+5}\right).$

33. $\dfrac{(a+c)^2-x^2}{a^2-ax+ac}\times\dfrac{a}{(a+x)^2-c^2}\div\dfrac{ac-cx-c^2}{(a-c)^2-x^2}.$

34. $\dfrac{3x^2+x-24}{6xy-20y}\times\dfrac{6x^2}{x^2-9}\times\dfrac{x-3}{3x-8}\div\dfrac{3x^2+9x}{3x^2-x-30}.$

35. $\dfrac{x^2-81}{x^2+25x+136}\times\dfrac{x^3+9x^2-136x}{x^3-729}\div\dfrac{x^2+x-72}{x^2+9x+81}.$

36. $\dfrac{6x^2-22x+20}{3x^2+4x+1}\times\dfrac{4x^2-5x+1}{4x^2-10x+4}\times\dfrac{6x^2-x-1}{12x^2-23x+5}.$

37. $\dfrac{x^2+x-6}{x^2-x-30}\times\dfrac{x^2+6x+5}{x^2-4x+4}\div\left(\dfrac{x^2+4x+3}{x^2-4x-12}\times\dfrac{3+x}{x-2}\right).$

38. $\dfrac{x+6+\dfrac{8}{x}}{1+\dfrac{2}{x}-\dfrac{8}{x^2}}.$

39. $\dfrac{x-3-\dfrac{1}{x-3}}{x-2-\dfrac{3}{x-4}}\times\dfrac{x-4-\dfrac{1}{x-4}}{x-5+\dfrac{3}{x-1}}.$

40. $\dfrac{\dfrac{a}{c^2}+\dfrac{c}{a^2}}{\dfrac{1}{a^2}-\dfrac{1}{ac}+\dfrac{1}{c^2}}.$

41. $\dfrac{\dfrac{a^3-x^3}{a^3+x^3}\times\dfrac{a+x}{(a-x)^2}\div\dfrac{a^2+x^2+ax}{a^2+x^2-ax}}{\dfrac{a^6+x^6}{a^6-x^6}\times\dfrac{a-x}{a^2+x^2}\div\dfrac{a^4+x^4-a^2x^2}{a^4+x^4+a^2x^2}}.$

TO THE TEACHER.—Teach 192 . . . . 214.

SIMPLE EQUATIONS.

1. $6x - 14 = x + 16.$
2. $12 - 4x = 24 - 7x.$
3. $5x - 11 = 13 - x.$
4. $3x - 24 = 9x - 42.$
5. $8x - 28 = x + 28.$
6. $11 - 2x = 46 - 7x.$
7. $27 - 4x = 12 - x.$
8. $4x + 15 = 29 - 3x.$
9. $5x + 17 = x + 19.$
10. $17 - 7x = 3x - 13.$
11. $7x - 36 = 36 - x.$
12. $6x - 12 = 16 - 8x.$
13. $15 + 8x = x + 18.$
14. $16 - 5x = 6x - 28.$
15. $9x + 26 = 36 - x.$
16. $6x + 17 = 22 + 4x.$
17. $24 - 2x = 16 - x.$
18. $4x + 17 = 20 - 2x.$
19. $7x - 24 = 16 - x.$
20. $17 - 8x = 5x - 22.$
21. $28 - 3x = 13 - x.$
22. $4x + 32 = 9x + 17.$
23. $5x - 14 = 2x + 7.$
24. $7x + 15 = 2x + 35.$
25. $18 - 7x = 3x - 42.$
26. $8x + 16 = 5x - 11.$
27. $6x + 13 = 2x + 17.$
28. $3x - 12 = 9x + 18.$
29. $8x - 15 = 3x + 20.$
30. $24 - 2x = 15 - 5x.$
31. $32 - 3x = 5x - 16.$
32. $4x + 14 = 2x + 15.$
33. $7x + 13 = 17 + 9x.$
34. $7x - 12 = 9x - 18.$
35. $14 + 8x = 3x + 16.$
36. $15 - 4x = 3x + 22.$
37. $9x - 19 = 44 + 2x.$
38. $18 - 3x = 2x + 15.$
39. $5x + 32 = 14 + 2x.$
40. $7x - 15 = 3x + 13.$
41. $6x + 16 = 3x + 26.$
42. $10 - 5x = 2x + 38.$
43. $5x + 34 = 7x + 25.$
44. $6x - 14 = 26 - 2x.$
45. $4x - 13 = 9x + 12.$
46. $4x + 16 = 21 - 3x.$
47. $3x + 43 = 13 + 8x.$
48. $4x + 14 = 29 + 6x.$

49. $(x+7)(2x-5)=(x+3)(2x-1)$.

50. $(3x-5)(4+2x)=(2+2x)(3x-4)$.

51. $(7-4x)(4x+2)=(8x+1)(4-2x)$.

52. $8(x+4)-6(5-x)=38x-4(3-x)$.

53. $(x-3)(7+x)-(x-4)(2+x)-8=0$.

54. $(x-4)(4+x)-(x-5)(5+x)-x=0$.

55. $3(x-2)-2(x-3)+4(x-5)-3=7$.

56. $5(x+3)-4(x-2)-3(2+x)-7=0$.

57. $5x-3(x-6)=2(8-x)-4(9+x)+6$.

58. $x(x+6)-(x-2)(x-3)-8(x+2)+17=0$.

59. $(5x+2)^2-(4x+3)^2-(3x-3)^2-7x-3=0$.

60. $2(3x-2)^2-2(6+x)+8(2x-1)-2(1-3x)^2=0$.

61. $4(5+6x)-2(3x-2)(1-2x)-3(2x-3)^2-20=0$.

62. $2x+4a=x+6a$.

63. $3a-4x=3-7x$.

64. $3x-3a=x+5a$.

65. $5x-2a=3x-2$.

66. $4x-5a=x-4a$.

67. $3x+5a=2+4x$.

68. $7a-3x=x-5a$.

69. $5x-4a=x-4b$.

70. $6a-6x=4a-x$.

71. $7x-3a=4a+7$.

72. $3x+9a=x+3a$.

73. $6x+2a=5a-x$.

74. $2a-5x=x+8a$.

75. $7x-6a=4x-3$.

76. $ax-4a=x-4b$.

77. $3c+ax=2b+6x$.

78. $4a+ax=5a-x$.

79. $3x-3b=ax-ab$.

80. $2c-bx=3c-x$.

81. $ax-5b=5a-bx$.

82. $ax-3a=9-3x$.

83. $3a+ax=bx+5a$.

84. $b^2-bx=a^2-ax$.

85. $bx-3b=2a+cx$.

86. $ax-a^3=b^3-bx$.

87. $ax-18=2a-9x$.

1. $\frac{2x}{3}+2\frac{1}{2}+\frac{x}{2}=x+\frac{3x+5}{10}$.

2. $\frac{a}{3}+x-\frac{x}{4}+\frac{a}{2}=a-\frac{x}{3}$.

3. $\frac{x}{4}-1\frac{1}{4}+x=\frac{6x-9}{3}-\frac{2x}{3}$.

4. $\frac{x}{b}-2-\frac{b}{a}+\frac{x}{a}-\frac{a}{b}=0$.

5. $\frac{x}{3}+\frac{4x+5}{7}-\frac{2x}{10}=x-6\frac{2}{3}$.

6. $a+\frac{4}{x}+b-\frac{3}{x}=\frac{5}{x}-b$.

7. $\frac{2x}{5}-3\frac{1}{5}-x=\frac{3x+1}{2}-\frac{x}{4}$.

8. $\frac{a}{5}-x+\frac{x}{3}+\frac{a}{3}=a-\frac{x}{5}$.

9. $\frac{5x+9}{8}-x+\frac{3x}{4}-1\frac{2}{3}=\frac{x}{3}$.

10. $\frac{a}{x}-b-1+\frac{c}{x}=\frac{b}{x}+a$.

11. $x+\frac{4x}{5}-3\frac{3}{5}+\frac{x}{4}=\frac{9x+5}{5}$.

12. $1-\frac{x}{a}-\frac{b}{a}+\frac{x}{b}=\frac{a}{b}-1$.

13. $\frac{x}{2}+4\frac{1}{3}+\frac{8x-5}{6}-\frac{4x}{9}=x$.

14. $\frac{a}{3}-\frac{1}{x}+\frac{2}{3}-\frac{a}{x}=\frac{5}{3}-\frac{2}{x}$.

15. $\frac{4x+5}{9}-\frac{x-5}{6}-\frac{x-8}{3}=2\frac{1}{3}+\frac{3+2x}{9}$.

16. $\frac{4x-3}{2}-x+\frac{x}{3}-\frac{x+8}{4}=8\frac{1}{2}-\frac{2x+3}{2}$.

17. $\frac{2x-4}{5}-\frac{x-8}{4}+\frac{x+6}{2}=2x-\frac{5x-12}{4}$.

18. $\frac{3x+8}{3}-\frac{x-6}{8}+7\frac{2}{3}=\frac{5x+15}{2}-\frac{x-3}{3}$.

19. $6\frac{3}{4}+\frac{3(x-6)}{2}-2(x-3)=\frac{x+7}{2}-\frac{5(x-2)}{4}$.

20. $\frac{4(x-2)}{3}-\frac{3(x+2)}{2}+9\frac{2}{3}=\frac{3x+8}{2}-\frac{2(x-3)}{3}$.

21. $9\frac{1}{2}-\frac{(x-3)2}{5}-(x+4)3=\frac{8x+5}{2}-\frac{(x+2)4}{5}$.

22. $\frac{8}{x-1}-\frac{30}{x^2-1}=\frac{6}{1+x}$.

23. $\frac{6}{x+a}=\frac{5}{x-a}$.

24. $\frac{x+2}{x-2}=\frac{4x+8}{x^2-4}-\frac{x-2}{2+x}$.

25. $\frac{a}{a-b}=\frac{a+b}{a-x}$.

26. $\frac{1-x}{x+1}=\frac{3x+4}{1-x^2}-\frac{x+3}{1-x}$.

27. $\frac{a+c}{x+2}=\frac{a-c}{x-2}$.

28. $\frac{8x+3}{x^2-9}-\frac{x+2}{3+x}=\frac{x+3}{x-3}$.

29. $\frac{x+3}{x-3}=\frac{a+c}{a-c}$.

30. $\frac{2-x}{x+2}=\frac{2x^2+2x}{4-x^2}-\frac{x+2}{2-x}$.

31. $\frac{a-x}{a+x}=\frac{x^2}{a^2-x^2}$.

32. $\frac{2x+1}{2x-1}-\frac{3x^2}{4x^2-1}=\frac{2x-1}{1+2x}$.

33. $\frac{4x+a}{6x+c}=\frac{2x-c}{3x-a}$.

34. $\frac{8}{x+1}-\frac{25}{(x+1)2}-5=\frac{x^2}{1+x}-x-\frac{5x}{x+1}$.

35. $\frac{3}{x-1}-4+\frac{4}{3(x-1)}=\frac{5}{2(x-1)}-x+\frac{x^2}{x-1}$.

36. $\frac{x-2}{3}-6-\frac{9}{3(x-3)}-x=\frac{x}{x-3}-\frac{2x^2}{3(x-3)}$.

37. $\frac{2+x}{x-2}-\frac{7x+86}{2(x^2-4)}-7=\frac{x-2}{2+x}-\frac{9x-1}{2(x-2)}-2\frac{1}{2}$.

38. $\frac{1+x}{x-1}-5-\frac{x^2+5}{3(x^2-1)}=\frac{x-1}{1+x}-3\frac{1}{3}-\frac{10x}{(x-1)2}$.

39. $\frac{2}{x-3}-3+\frac{17}{(x+2)12}=\frac{x}{x-3}-3\frac{1}{2}-\frac{2x+9}{4(x+2)}$.

40. $\frac{x}{x+1}-\frac{x}{x+2}+\frac{9}{4(x+1)}=\frac{11}{(x+2)2}-\frac{5}{(x+1)3}$.

41. $\frac{x}{a-c}-\frac{2}{a+c}-\frac{4ax-13a}{4(a^2-c^2)}=\frac{5}{(a-c)3}-\frac{x}{2(a+c)}$.

TO THE TEACHER.—Teach 215 . . . . 218.

1. $\dfrac{8x+7}{12}-1\tfrac{1}{4}-\dfrac{8x-9}{2x+5}=\dfrac{2x-5}{3}-\dfrac{4x+2}{5+2x}.$

2. $\dfrac{9x+4}{15}-\dfrac{3x+2}{3x-4}+2\tfrac{1}{3}=\dfrac{3x+3}{5}-\dfrac{2x-5}{3x-4}.$

3. $\dfrac{8x+5}{4}-2\tfrac{1}{2}=\dfrac{5x+2}{2x+3}+\dfrac{6x-5}{3}-\dfrac{3x+6}{2x+3}.$

4. $\dfrac{6x-5}{8}-\dfrac{4x+5}{3x+2}+3\tfrac{1}{3}=\dfrac{3x+9}{4}-\dfrac{2x+7}{2+3x}.$

5. $\dfrac{4x+5}{5x-3}+\dfrac{3x-8}{3}-\dfrac{4x+9}{4}=\dfrac{7x+3}{5x-3}-5\tfrac{1}{2}.$

6. $\dfrac{6x+5}{4}-\dfrac{5x-3}{2x-1}-\dfrac{1}{2}x=\dfrac{3x+2}{3}-\dfrac{4x-4}{2x-1}.$

7. $\dfrac{5x+2}{5}-\dfrac{2}{3}x-\dfrac{3x+2}{2x+3}=\dfrac{2x-3}{6}-\dfrac{2x-5}{3+2x}.$

8. $\dfrac{3x+2}{4}+\dfrac{4+3x}{4-5x}-\dfrac{2}{3}x=\dfrac{2x+9}{24}-\dfrac{2-4x}{4-5x}.$

9. $\dfrac{8x+5}{9}-\dfrac{4}{5}x-\dfrac{3x-a}{2x-a}=\dfrac{4x-2}{45}-\dfrac{2x-2a}{2x-a}.$

10. $\dfrac{6x+3}{4}-\dfrac{2+8x}{3x-1}-\dfrac{3x-5}{3}=\dfrac{x+3}{2}-\dfrac{5x+6}{3x-1}.$

11. $\dfrac{3x-4}{6}-\dfrac{5x+7}{1-3x}+\dfrac{3x+2}{1-3x}=\dfrac{2x-5}{4}-\dfrac{4x+4}{1-3x}.$

12. $\dfrac{6x+7}{5a}-\dfrac{5x+8}{14x}-\dfrac{8x+7}{7a}=\dfrac{2x+9}{35a}-\dfrac{3x+4}{14x}.$

13. $\dfrac{5x-4}{3}-\dfrac{7x-2}{4x-5}-\dfrac{7x-6}{5}=\dfrac{4x-7}{15}-\dfrac{4x+7}{4x-5}.$

14. $\dfrac{2x-5}{2a}+\dfrac{2x-3}{3x}-\dfrac{2x-4}{3a}=\dfrac{4x-5}{12a}-\dfrac{5x-3}{6x}.$

TO THE TEACHER. — Teach 219 . . . . 224, also 225 . . . . 236, as problems are found illustrating these suggestions.

PROBLEMS INVOLVING SIMPLE EQUATIONS.

1. The larger of two numbers is 4 times the smaller, and the sum of the numbers is 85. Find the numbers.

2. Seven times the smaller of two numbers equals 5 times the greater, and their sum is 84. Find the numbers.

3. The sum of two numbers is 48, and 5 times the less exceeds 3 times the greater by 32. Find the numbers.

4. A man bought two horses for three hundred sixty dollars, paying eighty dollars more for the better one than he paid for the other. Find the cost of each.

5. A man left one-half of his property to his wife, one-third to his daughter, and the remainder, which was $ 5500, to his son. How much did the wife receive?

6. A man paid two hundred twenty-five dollars for a horse and carriage, paying $ 85 less for the carriage than he paid for the horse. Find the cost of each.

7. Find two numbers which are to each other as three to seven, and whose sum is 140.

8. Find two numbers differing by thirty-six whose sum is equal to twice their difference.

9. A man has four children the sum of whose ages is 64 years, and the common difference of their ages is twice the age of the youngest. Find the age of the oldest.

10. A boy ate $\frac{1}{2}$ of his oranges and gave away $\frac{1}{3}$ of them. The difference between the number he ate and the number he gave away was four. How many did he have?

11. Divide 28 into two parts such that six times the less shall exceed three times the greater by 24.

12. A can build a wall in $4\frac{1}{2}$ days, and B can build it in 3 days. In how many days can both build it?

13. A can do a piece of work in m days, and B can do it in n days. In how many days can both do it?

14. A man saved \$2160 in two years. If he saved two and three-fifths times as much the second year as he saved the first, how much did he save each year?

15. A man bequeathed his property, amounting to \$22,600, to his wife, son, and daughter. The son received \$600 more than the daughter, and \$1000 less than the wife. How much did the wife receive?

16. A collection, amounting to \$6.40, consisted of dimes, 5-cent pieces, and 2-cent pieces. There were three times as many 2-cent pieces as 5-cent pieces and twice as many 5-cent pieces as dimes. How many coins were there?

17. A man paid \$9.55 for a hat and coat. The coat cost \$4.05 more than the hat. Find the cost of each.

18. Eight men agreed to share equally in buying a boat, but two being unable to pay their share, the others had each to pay \$20 more. How much did the boat cost?

19. A man left an estate of \$26,000 to be divided among his wife, two sons, and two daughters. The wife received twice as much as each daughter, and each son received $\frac{1}{2}$ as much as each daughter. How much did each receive?

20. A grocer mixed tea worth 60 cents a pound with tea worth 40 cents a pound in such proportions that the mixture, weighing 120 pounds, was worth fifty-six dollars. How many pounds of each kind did he take?

21. The sum of two numbers is 66, and their difference is 28. Find the numbers.

22. A man is three times as old as his son, but 5 years ago he was 4 times as old. Find the age of each.

23. A can do a piece of work in five days; B can do it in six days; and C can do it in ten days. In how many days can all do it, working together?

24. A and B can do a piece of work in 12 days, and A can do it in 20 days. In how many days can B do it?

25. A man spent one-sixth of his money for a suit of clothes and one-fourth of it for a watch, and had eighty-four dollars left. How much did he spend?

26. The sum of two numbers is fifty-eight; and if the greater be divided by the less, the quotient will be 4 and the remainder three. Find the numbers.

27. A's age is to B's as 4 to 5, and the sum of their ages is 117 years. Find the age of each.

28. A man bought land for $720. He gave his son twenty-four acres and sold one-half the remainder at cost for three hundred dollars. How many acres did he buy?

29. The value of a man's horse and carriage is $365; the value of his horse and harness is $280; the value of the carriage and harness is $195. Find the value of all.

30. Eight boys and 18 men earn two hundred ninety-four dollars a week. If each man earns five times as much as each boy, how much do the 8 boys earn per week.

31. The sum of two numbers is a, and their difference is b. What are the numbers?

32. Divide 32 into two such parts that the sum of twice the less and 5 times the greater shall be 118.

33. Four times a certain number is 54 more than twice the number. Find the number.

34. The sum of two numbers is a, and the greater is b times the less. Find the larger number.

35. If A can do one-half of a piece of work in six days and B can do the whole of it in eight days, in how many days can both do the work?

36. A man invested a certain sum at 5 per cent and twice as much at 6 per cent. His annual income from both investments was $765. How much did he invest?

37. A man left $\frac{2}{5}$ of his estate to his wife and $\frac{1}{4}$ of it to his daughter. The wife received $4200 more than the daughter. How much did each receive?

38. A boy has $9 in quarters and dimes, and he has 5 times as many dimes as quarters. How many coins has he?

39. The difference of two numbers is thirty-two; and if the greater be divided by the less, the quotient will be 5 and the remainder 4. Find the numbers.

40. A man divided $5000 among his five sons so that each one received two hundred dollars less than his next older brother. How much did the youngest son receive?

41. Three men earned a certain sum of money. A and B earned $140; A and C earned $125; and B and C earned $115. How much did they all earn?

42. A had nine acres of land more than B; but A sold B twenty-three acres, when he had only half as many acres as B. How many acres had each at first?

43. A can do a piece of work in seven and one-half days; B, in five days; and C, in three and one-third days. In how many days can all do the work?

44. At what rate per annum will \$7500 yield \$450 interest in one year and six months?

45. A man spends one-fifth of his annual salary for board, one-eighth for clothes, one-fourth for other expenses, and saves \$850. How much are his annual expenses?

46. Divide the number two hundred ten into two parts such that the quotient of the greater divided by the less is four and the remainder ten.

47. A's age is to B's age as six to five, but twenty-eight years ago their ages were to each other as two to one. How old is each at the present time?

48. In a regiment of soldiers there is one officer to every thirty men; and if the whole number of men be divided by the number of officers less ten, the quotient will be 62. Find the number of men in the regiment.

49. A boy earns 75 cents a day less than his father, and in twelve days the father earns \$12 more than the son earns in nine days. How much do both earn per day?

50. Divide ten dollars into two parts so that there may be twice as many dimes in the first part as there are five-cent pieces in the second part.

51. The difference between two numbers is eleven; and if three be added to the greater, the sum will equal three times the less. Find the numbers.

52. The sum of the third, fourth, and eighth parts of a number is thirty-four. Find the number.

53. A is three times as old as B, but in 16 years he will be only twice as old. Find the age of each.

54. A can do a piece of work in 12 days; A and C, in nine days; and A and B, in six days. In how many days can B and C together do the work?

55. What sum must be invested at $5\frac{1}{2}$ per cent to produce a quarterly income of $4125?

56. A man bought 25 sheep, paying $2.25 a head for some of them, and $3.50 a head for the others. The average cost was $2.80. How many did he buy at $2.25?

57. A and B are fifty-two miles apart. They travel toward each other, A four miles an hour and B two miles an hour, and B sets out two hours before A. How many miles will A have traveled when they meet?

58. A merchant mixes a pounds of tea worth d cents a pound, b pounds worth e cents a pound, and c pounds worth f cents a pound. Find the value of the mixture per pound.

59. There are 85 people in a railway car. There are five more men than women and children, and ten more women than children. How many are there of each?

60. A had three times as many sheep as B. Each sold to C half his flock, and A sold 40 to B, when A and B had the same number. How many did each have at first?

61. At what time between 4 and 5 o'clock are the hands of a clock together?

62. At what time between 2 and 3 o'clock are the hands of a clock at right angles to each other?

63. At what time between 3 and 4 o'clock are the hands of a clock opposite each other?

64. A can do a piece of work in 10 days, which B can do in 12 days, and with C's help they can do it in four days. In how many days can C do the work?

65. What is the distance between two cities, if an express train, running forty miles an hour, can go from one city to the other in two hours less time than a freight train, running twenty-four miles an hour?

66. What principal at r per cent interest will amount to a dollars in b years?

67. Two numbers are to each other as three to four; but if fifty be subtracted from each number, the remainders will be to each other as one to two. Find the numbers.

68. A man bought a horse, carriage, and harness for \$390. He paid \$60 more for the horse than for the other two, and the carriage cost \$5 more than three times as much as the harness. Find the cost of the horse.

69. A man bought some cows at thirty dollars a head. If he had bought two more for the same money, each one would have cost five dollars less. How many did he buy?

70. A merchant divided a sum of money among his four clerks. He gave the first $\frac{1}{3}$ of the whole; the second, forty dollars; the third, $\frac{1}{5}$ of the whole; and the fourth, thirty dollars. How much did he give to all?

71. A, B, and C together earn \$4200. A's salary is $\frac{2}{3}$ of B's and \$350 less than C's. Find C's salary.

72. At what time between 8 and 9 o'clock are the hands of a clock together?

73. A is 48 years old, and B is $\frac{2}{3}$ as old. How many years have elapsed since B was half as old as A?

74. A and B can do a piece of work in eight days; A and C, in seven days; and A alone, in twelve days. In how many days can B and C do the work?

75. Three-fifths of a certain principal was invested at 4 %, and the remainder at 5 %. The annual income from both investments was \$550. Find the sum invested.

76. A boy has ten dollars in half-dollars and five-cent pieces, there being in all fifty-six coins. How many coins has he of each kind?

77. A man going from a certain town traveled at the ra‹ of 4 miles an hour. Five hours afterward a horsema‹ going at the rate of 6 miles an hour, was sent after hi‹ How far did the latter travel to overtake the former?

78. A merchant sold silk for a dollars and gained b pe‹ cent. How much did he pay for it?

79. A lady bought 16 yards of silk; but if she had bought four yards more for the same money, it would have cost 25 cents a yard less. How much did it cost?

80. D is 4 years older than C; C is 3 years older than B; B is 2 years older than A; and in five years the sum of their ages will be 92 years. Find D's age.

81. One of two numbers is three times the other. If 18 be subtracted from the greater, and the less from 38, the remainders will be equal. Find the numbers.

82. The sum of two numbers is a, and m times the less equals n times the greater. Find the numbers.

83. A and B can do a piece of work in five days, which A and C can do in six days, and which B and C can do in $7\frac{1}{2}$ days. In how many days can all do the work?

84. At what time between 8 and 9 o'clock are the hands of a clock at right angles to each other?

85. At what rate per annum will c dollars yield a dollars interest in b years?

86. Four hours after a train left Albany, a second train set out to overtake the first in six hours. To accomplish this, it was necessary for the second train to run sixteen miles an hour faster than the first. How many miles per hour did the first train run?

87. A's money is to B's as three to seven, and together they have $15,000. How much money has each?

88. A farmer bought sheep at three dollars a head and had \$ 18 left; but if he had bought them at three dollars seventy-five cents a head, he would have needed 75 cents more to pay for them. How many sheep did he buy?

89. If it costs the same, at \$ 2 a yard, to inclose a square court with a fence as to pave it at 40 cents a square yard, what are the dimensions of the court?

90. A man owed \$ 92. He sold wheat at \$ .75 a bushel and corn at \$ .40, selling the same number of bushels of each. If he received just money enough to pay the debt, how many bushels of grain did he sell?

91. Find the number whose double diminished by seventeen is as much less than forty-eight as twenty-two is less than the number itself.

92. A father is forty-five years old, and his son is one-third as old. In how many years will the son be one-half as old as his father?

93. At what times between 4 and 5 o'clock are the hands of a clock at right angles to each other?

94. The length of a rectangle exceeds its breadth by nine inches. If the length were diminished four inches and the breadth increased three inches, the area would remain the same. Find the dimensions of the rectangle.

95. A man was hired for forty days at the rate of \$ 2.75 a day and his board, and for every day he might be idle he was to pay \$ .75 for his board. At the end of the time he received \$ 89. How many days did he work?

96. After A had been traveling four hours at the rate of seven miles in two hours, B set out at the rate of eleven miles in two hours to overtake A. In how many hours from the time B started did he overtake A?

97. A lady bought 42 yards of silk for \$48, paying \$1 a yard for part of it and \$1.25 a yard for the rest. How many yards of each kind of silk did she buy?

98. By selling tea at a cents a pound, a grocer lost b per cent. How much did he pay for the tea?

99. Two men engage in business. A invests $\frac{3}{4}$ as much as B. The first year, A gains \$1500, which he adds to his capital, and B loses $\frac{1}{6}$ of his money. The next year, A loses $\frac{1}{10}$ of his money, and B gains \$400, when they have equal amounts. How much had each at first?

100. A lady bought a hat and a pair of shoes, paying three-fourths as much for the hat as she paid for the shoes. If she had paid \$2 less for the shoes and \$1.50 more for the hat, the hat would have cost $\frac{3}{2}$ times as much as the shoes. How much did she pay for both?

101. The difference between the sixth and eighth parts of a number is six. Find the number.

102. Between what hours are the hands of a clock together? Form the equations for finding the times.

103. A man is 24 years older than his son, but two years ago he was 4 times as old. Find the father's age.

104. A can do a piece of work in 8 days, and B can do it in 12 days. If B works two days alone, in how many days can he complete it with A's assistance?

105. A boy bought a number of apples at the rate of three for five cents, and sold them at the rate of five cents for two, gaining \$1. How many did he buy?

106. A general arranged part of his men in a solid square and had 40 men left; but attempting to add one man to each side of the square, he found that he needed 41 men to complete the square. How many men had he?

107. A man invests one-third of his money in 3 per cent bonds, one-fourth in $4\frac{1}{2}$ per cent bonds, and the remainder in 5 per cent bonds. His annual income from the whole investment is \$1010. Find the whole sum invested.

108. A merchant has tea worth 30 cents a pound and some other worth 50 cents a pound. How many pounds of each must he take to mix 60 pounds worth \$.46 a pound?

109. Divide a into two parts such that the quotient of the greater divided by the less is b and the remainder c.

110. A man bought two horses and a carriage, paying \$175 for the carriage. The better horse and carriage cost \$50 less than three times as much as the poorer horse, and the poorer horse and carriage cost twice as much as the better horse. Find the cost of the horses.

111. At what times between 5 and 6 o'clock are the hands of a clock twenty minutes apart?

112. The sum of the third and fourth parts of a certain number exceeds eight times the difference between the fifth and sixth parts by 38. Find the number.

113. A is fifty-five years old, and B is three-fifths as old. How many years have elapsed since A was two and four-sevenths times as old as B?

114. A can do half as much work as B; B can do $\frac{1}{3}$ as much as C; and together they can complete a piece of work in 8 hours. In what time can each complete it?

115. What sum must be invested at a per cent to give a semi-annual income of c dollars?

116. A and B are eighty-two miles apart. They set out at the same time and travel toward each other. A travels at the rate of ten miles in three hours, and B at the rate of seven miles in two hours. How many miles will each have traveled when they meet?

117. A mason received \$ 3.50 a day for his labor and paid \$ .75 a day for his board. At the end of thirty days he had saved \$ 54.50. How many days did he work?

118. A man walking one and a half miles an hour set out from a certain town, and two hours after another man walking three miles an hour set out to overtake him. How many miles did the second walk to overtake the first?

119. A man bought sheep at \$ 2.75 a head and had twenty-five dollars left; but if he had bought the same number at \$ 3.25 a head, he would have had only five dollars left. How many sheep did he buy?

120. A young man spends one-fourth of the money he has in bank at the beginning of each year, and adds to it during each year an annuity of \$ 6000. At the beginning of the fourth year he has \$ 20,625 to his credit in the bank. How much did he have in the bank at first?

121. Divide 88 into two parts such that $\frac{3}{4}$ of the less shall be equal to three-sevenths of the greater.

122. The width of a room is $\frac{5}{7}$ of its length. If the length were 3 feet less and the width 3 feet more, the room would be square. Find the dimensions of the room.

123. A cistern can be filled by two pipes in four hours and six hours respectively, and can be emptied by a third in twelve hours. In what time will the cistern be filled, if all three pipes are running together?

124. At what times between 1 and 2 o'clock are the hands of a clock at right angles to each other?

125. A boy bought some peaches at the rate of two for a cent and as many more at the rate of two cents for three. He sold them all at the rate of four for five cents and gained \$ 1.60. How many did he buy?

126. A man received an annual income equal to $\frac{1}{2}$ the sum invested. He invested the income, less \$4000 for annual expenses, at the same rate. At the end of 3 years he had \$21,500 invested. Find his original investment.

127. In how many years will a dollars amount to b dollars at c per cent interest?

128. A man bought a suit of clothes for \$24 and paid for it in two-dollar bills and fifty-cent pieces, giving twice as many coins as bills. How many bills did he give?

129. A and B have \$96. A buys sheep at \$4 a head and has \$12 left, and B buys the same number at \$5 a head and has the same sum left. How many does each buy?

130. A lady bought a cloak, a dress, and a hat, paying \$9 for the hat. The dress and hat cost $\frac{3}{4}$ as much as the cloak, and the cloak and hat cost \$7 more than twice as much as the dress. How much did she pay for all?

131. Divide 84 into two parts such that two-fifths of the greater shall exceed six-sevenths of the less by 16.

132. A's age is to B's age as 4 to 7, and the sum of their ages is 66 years. Find the age of each.

133. At what times between 6 and 7 o'clock are the hands of a clock at right angles to each other?

134. A can do a piece of work in 24 days, which B can do in 20 days. A begins the work, but after a time he is relieved by B, and the work is completed in $21\frac{1}{2}$ days from the beginning. How many days did each work?

135. If 1024 men be arranged in a hollow square 8 men deep, how many men will there be in the outside line?

136. Divide \$8000 into two parts such that the interest of the greater part for 4 years at 3% shall be the same as the interest of the smaller part for 5 years at 4%.

137. A man can row a miles an hour on still water. How far down a stream, whose current is b miles an hour, can he row so that he may go and return in c hours?

138. A wheelman who rides 32 miles an hour is 45 minutes in advance of a second, who rides 40 miles an hour. In how many hours can the second overtake the first?

139. The number of girls in a school is to the number of boys as five to three; but if there were fifteen more girls and fifteen less boys, the ratio would be as two to one. How many girls are there in the school?

140. A man bought two farms, one of which contained 40 acres less than 3 times as much as the other. He paid $10 an acre for the smaller farm and $4 an acre less for the other. He kept 10 acres of each farm and sold the remainder at $1 an acre above cost, receiving for it $100 more than the whole cost. How many acres did he buy?

141. Divide seventy-two into two parts such that the excess of the greater over thirty shall be four times the excess of thirty over the less.

142. Between what hours are the hands of a clock opposite each other? Form the equation for each solution.

143. The length of a rectangle is $3\frac{1}{2}$ times its width. If each dimension were two inches less, the area would be diminished 122 square inches. Find the length.

144. A, B, and C can together complete a piece of work in thirty days. A does three-fourths as much as B, and B does four-fifths as much as C. In how many days can each complete the work alone?

145. A man bought some eggs at thirty cents a dozen. He sold three-eighths of them at the rate of thirty cents for nine, and the remainder at the rate of fifteen for fifty cents. He gained $2. How many eggs did he buy?

146. If a regiment of troops be arranged in a solid square with a certain number of men on a side, there will be 232 men left; but if four men be added to each side of the square, there will be just enough men to complete the square. How many men are there in the regiment?

147. A merchant starts in business with \$8000 capital. His profits are equal to $\frac{1}{4}$ of his capital. If he adds his profits, less a fixed sum for annual expenses, to his capital, at the end of 3 years his capital will be increased to \$11,050. How much are his annual expenses?

148. A man has \$6.56 in dollars, dimes, and cents. He has $\frac{3}{8}$ as many cents as dimes and four times as many cents as dollars. How many pieces of money has he?

149. A has \$92 more than B. A buys land at \$8 an acre and has \$180 left, and B buys 8 acres less at \$12 an acre and has \$24 left. How many acres does each buy?

150. A man earns three times as much per day as his older son and five times as much as his younger son. The father worked twenty-four days, the older son twenty days, and the younger son sixteen days, and all earned \$127. How much do all earn per day?

151. A father is forty-two years old, and his son is three-sevenths as old. In how many years will the son be three-fifths as old as his father?

152. A tank can be filled by two pipes in 40 minutes and one hour 20 minutes respectively, and can be emptied by a third in two hours. In what time will the tank be filled, if all the pipes are running together?

153. A man was hired to do some work. He was to receive \$3 a day for every day he worked and forfeit \$.75 for every day he was idle. He worked 3 times as many days as he was idle and received \$66. How many days did he work?

154. At what times between 12 and 1 o'clock are the hands of a clock at right angles to each other?

155. A fox is pursued by a hound. The fox makes 4 leaps while the hound makes 3, but 3 leaps of the hound are equal to 5 of the fox. The fox has a start of 30 of her own leaps. How many leaps must the hound make to catch the fox?

156. What sum must be invested at four per cent to give a semi-annual income of $1680?

157. A pedestrian walked a certain distance at the rate of $2\frac{1}{2}$ miles an hour. He rested an hour at the end of his journey and returned at the rate of 3 miles an hour. If he was out 12 hours, how far did he walk?

158. A and B are fifty-three miles apart. They travel towards each other, A $2\frac{1}{2}$ miles an hour, and B $1\frac{1}{2}$ miles an hour. If B starts two hours later than A, how far will each have traveled when they meet?

159. A lady bought silk at eighty cents a yard and had $2 left; but if she had bought it at ninety-five cents a yard, she would have needed twenty-five cents more to pay for it. How many yards of silk did she buy?

160. If 18 pounds of copper weighs 2 pounds less in water than in air, and 21 pounds of tin weighs 3 pounds less in water than in air, how many pounds of each are there in a mixture of the two metals, which weighs 450 pounds in air and 392 pounds in water?

161. The side of a square is four inches less than the length and three inches more than the side of an equivalent rectangle. Find the area of the rectangle.

162. A can do a piece of work in 24 days, which B can do in 20 days. B begins the work, but after a time he is relieved by A, who finishes the work, working two days more than B. How many days does each work?

163. A is 20 years older than B, but in two years A will be only twice as old as B. Find the age of each.

164. At what times between 10 and 11 o'clock are the hands of a clock at right angles to each other?

165. A man bought some oranges at the rate of two for five cents and twice as many more at the rate of five cents for three. He sold them all at thirty-five cents a dozen and gained \$1.75. How many did he buy?

166. A merchant's profits are equal to $\frac{1}{8}$ of his capital. If he adds the profits, less \$3000 for annual expenses, to his capital, at the end of three years his capital will be increased to \$21,800. Find the original capital.

167. At what rate per annum will a dollars amount to b dollars in c years?

168. A merchant sold eighty pounds of tea, part of it at forty cents a pound and the rest at sixty cents a pound. The average price was forty-eight cents a pound. How many pounds did he sell at each price?

169. Two boys had equal sums of money. One bought some oranges at 5 cents apiece and had 3 cents left; the other bought $1\frac{1}{8}$ times as many oranges at \$.36 a dozen and had 12 cents left. How much money had each?

170. A man spent half a dollar more than half his money. Then he spent half a dollar more than half of what remained. A third time he did the same, when he had nothing left. How much money had he at first?

171. A, B, and C can complete a piece of work in thirty hours, A doing half as much again as B, and B doing two-thirds as much again as C. In how many hours can each one complete the work alone?

172. At what times between 9 and 10 o'clock are the hands of a clock twenty minutes apart?

173. Eight years ago A was three times as old as B, but in eight years he will be only twice as old. Required the age of each at the present time.

174. A hare is pursued by a dog. The hare makes 5 leaps while the dog makes 4, but 5 leaps of the dog are equal to 7 of the hare. The hare has a start of 48 of her own leaps. How many leaps will the hare have made when caught?

175. An officer arranged part of his men in a solid square and had 80 men left; but upon attempting to form them into a column with 15 men more in front and 8 men less in depth, he found that he lacked 80 men to complete the column. How many men did he have?

176. A man invests $10,000, part of it at six per cent, and the rest at four per cent. The interest on the former for four years is the same as that on the latter for two years. Find the sum invested at four per cent.

177. A man can row 5 miles an hour on still water. How far down a river, whose current is 2 miles an hour, can he row so that he may go and return in ten hours?

178. A and B set out at the same time from two places and travel toward each other, A traveling $3\frac{1}{2}$ miles an hour and B $4\frac{1}{4}$ miles an hour. When they meet, B has traveled 12 miles more than A. How far apart were they?

179. A man bequeathed his property to his son and daughter, leaving the son fifteen hundred dollars more than $\frac{4}{9}$ of the whole. The son's share was to the daughter's share as 19 to 17. Find the amount of the estate.

180. A man paid thirty dollars for a hat and coat. The difference in the cost of the two was six times the cost of the hat. Find the cost of the coat.

181. Eighteen years ago a man was two-fifths as old as he will be in 12 years. What is his age at present?

182. At an election, two candidates together received 4229 votes, and the candidate elected received a majority of 521. How many votes did each receive?

183. A cask contains 12 gallons of wine mixed with 18 gallons of water, and another cask contains 18 gallons of wine mixed with six gallons of water. How many gallons must be drawn from each to make a mixture of fourteen gallons of wine and fourteen gallons of water?

184. At what times between 7 and 8 o'clock are the hands of a clock at right angles to each other?

185. The width of a rectangular piece of paper is 6 inches more than half its length; and if a strip 3 inches wide were cut off from the four sides, it would contain 360 sq. in. Find the dimensions of the paper.

186. A tank can be filled by two pipes in eighteen minutes. After the first has been running by itself for six minutes, the second is turned on, and the tank is filled in fourteen minutes more. In how many minutes can it be filled by each pipe separately?

187. I bought some oranges at the rate of five for three cents. I sold one-third of them at the rate of five cents for three, and the remainder at one cent apiece, gaining one dollar twelve cents. How many did I buy?

188. An officer can form his men into a hollow square six men deep, and also into a hollow square eight men deep. There are eight more men in the front of the first square than in the second. How many men has he?

189. A farmer sold wheat at 80 cents a bushel, rye at 75 cents a bushel, and corn at 60 cents a bushel, receiving for it all one hundred eighteen dollars. He sold ten bushels less of rye than of wheat, and twice as many bushels of corn as of rye. How many bushels of grain did he sell?

190. Of two consecutive numbers, $\frac{1}{4}$ of the less exceeds $\frac{1}{5}$ of the greater by two. Find the numbers.

191. A has twenty-five dollars less than B. A buys twenty-four sheep and has \$18 left, and B buys twenty-one more than A at \$1 less per head and has \$25 left. At what price per head did each man buy his sheep?

192. A man invests \$12,750. His income is equal to one-third of the investment. If he invests his income, less a fixed sum for annual expenses, at the same rate, at the end of three years his investment will amount to twenty-two thousand dollars. Find his annual expenses.

193. One boy takes from a basket one more than half the peaches in it, a second boy takes one more than half the remainder, and a third takes four more than half of those that still remained, and there were none left. How many peaches were there in the basket at first?

194. At what times between 11 and 12 o'clock are the hands of a clock twenty-five minutes apart?

195. B's age is to A's as 5 to 8, and the difference between their ages is 18 years. Find the age of each.

196. A and B can complete a piece of work in $2\frac{2}{3}$ days, A and C in three days, and B and C in $4\frac{4}{5}$ days. In how many days can each do the work alone?

197. A fox is pursued by a hound. The fox makes six leaps while the hound makes five, but two leaps of the hound are equal to three of the fox. The fox has a start equal to sixty of the dog's leaps. How many leaps will each have made when the fox is caught?

198. A farmer hired a laborer for a year, agreeing to pay him one hundred fourteen dollars and a suit of clothes. He worked only nine months and was entitled to receive \$81 and the clothes. Find the value of the clothes.

199. A man had two farms of equal size. He sold 20 acres from one and 60 acres from the other, when he found that there remained $\frac{5}{7}$ as many acres in one as in the other. How many acres were there in each farm at first?

200. A man invested his fortune. The annual interest at 5 per cent on \$ 700 more than one-fourth of it exceeds by \$ 40 the interest at 4 per cent on \$ 500 less than one-third of it. Find the sum invested at 5 %.

201. A man set out at 6 A.M. and drove a certain distance at the rate of 5 miles an hour. He returned on foot, walking 2 miles an hour, and reached home at 5.30 P.M., having rested one hour at noon. How far did he walk?

202. A railway train leaves a station, and 40 minutes after, another train, which runs 30 miles an hour, starts out and overtakes the first in 2 hours and 40 minutes. How many miles per hour does the first train run?

203. A boy bought some oranges at a cents apiece and had b cents left; but if he had had c cents more, he could have bought the same number of better ones at d cents apiece. How many oranges did he buy?

204. If a pound of tin weighs $\frac{32}{37}$ of a pound in water, and a pound of lead weighs $\frac{21}{23}$ of a pound in water, how many pounds of each metal are there in a mixture of 120 pounds, if it weighs only 106 pounds in water?

205. A man bequeathed his entire estate to his three sons as follows: to his eldest son he leaves \$ 1000, together with $\frac{1}{4}$ of what remains; to the second son he leaves \$ 2000, together with $\frac{1}{4}$ of what remains after the eldest son's share and \$ 2000 have been deducted; to the third he leaves \$ 3000, together with $\frac{1}{4}$ of what remains after the portions of the two other sons and \$ 3000 have been deducted. Find the amount of the estate.

TO THE TEACHER.—Teach 237 . . . . 245.

SIMULTANEOUS EQUATIONS.

ELIMINATION BY ADDITION OR SUBTRACTION.

1. $\begin{cases} 2x+3y=7 \\ 3x-\ \ y=5 \end{cases}$

2. $\begin{cases} 3x-2y=-5 \\ 2x+\ \ y=-8 \end{cases}$

3. $\begin{cases} 4x-3y=7 \\ 2x-2y=2 \end{cases}$

4. $\begin{cases} 5x+4y=-7 \\ 2x+\ \ y=-1 \end{cases}$

5. $\begin{cases} 6x+3y=4 \\ 8x-6y=2 \end{cases}$

6. $\begin{cases} 2x+3y=-4 \\ 3x+7y=-1 \end{cases}$

7. $\begin{cases} 4x-3y=16 \\ 7x-5y=29 \end{cases}$

8. $\begin{cases} 3x-5y=-32 \\ 7x-9y=-40 \end{cases}$

9. $\begin{cases} 5x+3y=38 \\ 9y-3x=15 \end{cases}$

10. $\begin{cases} 6x+4y=-22 \\ 5y+7x=-20 \end{cases}$

11. $\begin{cases} 7x-3y=29 \\ 8x-5y=32 \end{cases}$

12. $\begin{cases} 5x+3y=-34 \\ 8y+9x=-30 \end{cases}$

13. $\begin{cases} \dfrac{6x}{5}+\dfrac{3y}{2}=24 \\ \dfrac{7x}{2}-\dfrac{5y}{4}=25 \end{cases}$

14. $\begin{cases} \dfrac{6x}{23}-\dfrac{5y}{2}=-13 \\ \dfrac{4y}{3}-3x=-15 \end{cases}$

15. $\begin{cases} \dfrac{5x}{4}+\dfrac{2y}{5}=26 \\ \dfrac{7y}{3}-\dfrac{3x}{2}=11 \end{cases}$

16. $\begin{cases} \dfrac{3x}{7}+\dfrac{5y}{3}=-21 \\ \dfrac{7y}{6}+\dfrac{4x}{12}=-14 \end{cases}$

17. $\begin{cases} \dfrac{6x}{5}-\dfrac{3y}{4}=30 \\ \dfrac{5y}{4}-\dfrac{5x}{10}=25 \end{cases}$

18. $\begin{cases} \dfrac{4x}{9}-\dfrac{5y}{7}=-45 \\ \dfrac{6y}{35}+\dfrac{9x}{5}=-75 \end{cases}$

TO THE TEACHER. — Teach 246.

ELIMINATION BY SUBSTITUTION.

1. $\begin{cases} 4x - 8y = 8 \\ 3x + 6y = 9 \end{cases}$

2. $\begin{cases} 6x + 4y = -4 \\ 3x + 3y = -8 \end{cases}$

3. $\begin{cases} 3x - 3y = 6 \\ 4x - 5y = 3 \end{cases}$

4. $\begin{cases} 4x - 5y = -4 \\ 3x - 4y = -5 \end{cases}$

5. $\begin{cases} 4x + 2y = 4 \\ 8x + 5x = 3 \end{cases}$

6. $\begin{cases} 2x + 2y = -3 \\ 6x + 7y = -6 \end{cases}$

7. $\begin{cases} \frac{5x}{2} - \frac{3y}{2} = 1 \\ \frac{3x}{4} + \frac{2y}{3} = 7 \end{cases}$

8. $\begin{cases} \frac{3x}{7} - \frac{5y}{3} = -3 \\ \frac{9x}{14} - 3y = -6 \end{cases}$

9. $\begin{cases} \frac{2x}{5} + \frac{2y}{7} = 2 \\ \frac{5x}{3} + \frac{3y}{2} = 4 \end{cases}$

10. $\begin{cases} \frac{3x}{4} + \frac{3y}{2} = -6 \\ \frac{5x}{6} + \frac{7y}{5} = -4 \end{cases}$

11. $\begin{cases} 9x - 4y = 22 \\ 6x + 3y = 43 \end{cases}$

12. $\begin{cases} 4x - 7y = -35 \\ 3y - 7x = -22 \end{cases}$

13. $\begin{cases} 5x - 3y = 48 \\ 4y - 2x = 20 \end{cases}$

14. $\begin{cases} 7x + 6y = -70 \\ 4y + 5x = -40 \end{cases}$

15. $\begin{cases} \frac{4x}{3} + \frac{4y}{3} = 36 \\ \frac{6y}{5} - \frac{5x}{10} = 12 \end{cases}$

16. $\begin{cases} \frac{3x}{8} + \frac{3y}{10} = -21 \\ \frac{8y}{5} - \frac{7x}{4} = -22 \end{cases}$

17. $\begin{cases} \frac{7x}{11} - \frac{2y}{13} = 22 \\ \frac{4y}{3} - \frac{5x}{22} = 42 \end{cases}$

18. $\begin{cases} \frac{4x}{9} + \frac{7y}{6} = -43 \\ \frac{5y}{3} + \frac{5x}{3} = -15 \end{cases}$

TO THE TEACHER.—Teach 247 and 248.

ELIMINATION BY COMPARISON.

1. $\begin{cases} 2x+3y=38 \\ 3x+2y=37 \end{cases}$

2. $\begin{cases} 4x-5y=-27 \\ 3x-7y=-43 \end{cases}$

3. $\begin{cases} 4x+3y=20 \\ 8x+5y=34 \end{cases}$

4. $\begin{cases} 2x+5y=-32 \\ 2y-3x=-28 \end{cases}$

5. $\begin{cases} 7x-2y=13 \\ 2x+3y=43 \end{cases}$

6. $\begin{cases} 3x+2y=-33 \\ 5y-4x=-25 \end{cases}$

7. $\begin{cases} \dfrac{3x}{4}+\dfrac{3y}{2}=24 \\ \dfrac{7x}{3}-\dfrac{3y}{5}=22 \end{cases}$

8. $\begin{cases} \dfrac{4x}{7}+\dfrac{4y}{3}=-12 \\ \dfrac{2y}{5}-\dfrac{8x}{2}=-27 \end{cases}$

9. $\begin{cases} \dfrac{4x}{9}+\dfrac{5y}{7}=23 \\ \dfrac{7x}{2}-\dfrac{4y}{3}=35 \end{cases}$

10. $\begin{cases} \dfrac{4x}{3}-\dfrac{2y}{7}=-40 \\ \dfrac{5y}{4}+\dfrac{7x}{3}=-21 \end{cases}$

11. $\begin{cases} 8x-3y=17 \\ 6y+6x=43 \end{cases}$

12. $\begin{cases} 4x+9y=-22 \\ 6y-6x=-71 \end{cases}$

13. $\begin{cases} 7x-2y=11 \\ 7y-2x=29 \end{cases}$

14. $\begin{cases} 6x+8y=-14 \\ 4y-9x=-47 \end{cases}$

15. $\begin{cases} \dfrac{3x}{4}+\dfrac{2y}{5}=36 \\ \dfrac{7x}{4}-\dfrac{5y}{6}=31 \end{cases}$

16. $\begin{cases} \dfrac{7x}{2}+\dfrac{5y}{2}=-10 \\ \dfrac{4y}{3}-\dfrac{3x}{5}=-30 \end{cases}$

17. $\begin{cases} \dfrac{2x}{7}+\dfrac{5y}{6}=42 \\ \dfrac{3x}{2}-\dfrac{4y}{3}=15 \end{cases}$

18. $\begin{cases} \dfrac{3x}{7}+\dfrac{5y}{3}=-26 \\ \dfrac{9y}{2}-\dfrac{5x}{2}=-19 \end{cases}$

1. $\begin{cases} 5x + 6y = 17 \\ 6x + 5y = 16 \end{cases}$

2. $\begin{cases} 2x + 3y = 0 \\ 3x - 4y = a \end{cases}$

3. $\begin{cases} \frac{2}{5}x + 2y = 90 \\ 2x - \frac{7}{9}y = 62 \end{cases}$

4. $\begin{cases} 4x - 3y = a \\ 3x - 4y = a \end{cases}$

5. $\begin{cases} 4x + 3y = 23 \\ 7y + 7x = 21 \end{cases}$

6. $\begin{cases} ax + by = 2 \\ cx - dy = a \end{cases}$

7. $\begin{cases} \frac{2}{3}x + 3y = 42 \\ 3x - \frac{4}{5}y = 46 \end{cases}$

8. $\begin{cases} 2x - 3y = a \\ 3x + 5y = b \end{cases}$

9. $\begin{cases} \frac{3}{4}x - \frac{2}{3}y = 20 \\ 3x + 2y = 24 \end{cases}$

10. $\begin{cases} ax - by = 0 \\ bx - ay = c \end{cases}$

11. $\begin{cases} \frac{2}{3}x + 4y = 0 \\ 2y + \frac{7}{8}x = 13 \end{cases}$

12. $\begin{cases} ax + by = c \\ a'x - b'y = c' \end{cases}$

13. $\begin{cases} \frac{2}{3}x - \frac{1}{5}y = 17 \\ 5y - 2x = 15 \end{cases}$

14. $\begin{cases} ax - by = cd \\ bx + ay = ac \end{cases}$

15. $\begin{cases} cx + ay = 2ac \\ ax + cy = a^2 + c^2 \end{cases}$

16. $\begin{cases} x + y = a + c \\ ax - cy = c^2 - a^2 \end{cases}$

17. $\begin{cases} ax + cy = a^2 - c^2 \\ cx + ay = a^2 - c^2 \end{cases}$

18. $\begin{cases} ax - cy = a^2 - c^2 \\ ax + cy = (a + c)^2 \end{cases}$

19. $\begin{cases} a(x + y) + c(x - y) = 3 \\ a(x - y) + c(x + y) = 1 \end{cases}$

20. $\begin{cases} (a + c)x - (a - c)y = 2 \\ (a - c)x + (a + c)y = 2 \end{cases}$

21. $\begin{cases} (a + c)y - (a - c)x = ac \\ (a + c)x - (a - c)y = 3ac \end{cases}$

To the Teacher. — Teach 249.

22. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=7 \\ \dfrac{1}{x}-\dfrac{1}{y}=1 \end{cases}$

23. $\begin{cases} \dfrac{5}{x}-\dfrac{4}{y}=12 \\ \dfrac{4}{x}+\dfrac{3}{y}=22 \end{cases}$

24. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=a \\ \dfrac{1}{x}-\dfrac{1}{y}=b \end{cases}$

25. $\begin{cases} \dfrac{3}{x}+\dfrac{2}{y}=22 \\ \dfrac{5}{x}-\dfrac{7}{y}=16 \end{cases}$

26. $\begin{cases} \dfrac{a}{x}+\dfrac{b}{y}=c \\ \dfrac{a}{x}-\dfrac{b}{y}=d \end{cases}$

27. $\begin{cases} \dfrac{4}{x}-\dfrac{3}{y}=11 \\ \dfrac{8}{y}+\dfrac{3}{x}=39 \end{cases}$

28. $\begin{cases} \dfrac{a}{x}+\dfrac{b}{y}=\dfrac{d}{c} \\ \dfrac{a}{x}-\dfrac{b}{y}=\dfrac{c}{d} \end{cases}$

29. $\begin{cases} \dfrac{3}{2x}+\dfrac{2}{3y}=5 \\ \dfrac{5}{3y}-\dfrac{6}{4x}=2 \end{cases}$

30. $\begin{cases} \dfrac{x}{a}+\dfrac{y}{b}=2 \\ \dfrac{x}{c}-\dfrac{y}{d}=3 \end{cases}$

31. $\begin{cases} \dfrac{3}{ax}+\dfrac{2}{by}=3 \\ \dfrac{2}{ax}-\dfrac{3}{by}=2 \end{cases}$

32. $\begin{cases} \dfrac{x}{a}-\dfrac{y}{b}=\dfrac{1}{2} \\ \dfrac{x}{c}+\dfrac{y}{d}=\dfrac{1}{3} \end{cases}$

33. $\begin{cases} \dfrac{a}{bx}+\dfrac{b}{ay}=c \\ \dfrac{c}{dx}+\dfrac{d}{cy}=a \end{cases}$

34. $\begin{cases} \dfrac{x}{a}+\dfrac{y}{b}=c \\ \dfrac{x}{b}-\dfrac{y}{a}=d \end{cases}$

35. $\begin{cases} \dfrac{a}{bx}+\dfrac{b}{ay}=\dfrac{c}{d} \\ \dfrac{c}{ax}-\dfrac{d}{by}=\dfrac{a}{b} \end{cases}$

36. $\begin{cases} \dfrac{x}{a} - \dfrac{y}{b} = \dfrac{c}{d} \\ \dfrac{x}{b} + \dfrac{y}{a} = \dfrac{d}{c} \end{cases}$

37. $\begin{cases} \dfrac{4x+5y+5}{7x-2y+9} = \dfrac{5}{4} \\ \dfrac{3x+3y+3}{4x-3y+4} = \dfrac{5}{2} \end{cases}$

38. $\begin{cases} \dfrac{a}{x} - \dfrac{b}{y} = 1 \\ \dfrac{b}{x} + \dfrac{a}{y} = 2 \end{cases}$

39. $\begin{cases} \dfrac{x+5}{2} - \dfrac{y+2}{3} = 5 \\ \dfrac{y+8}{3} - \dfrac{x-3}{4} = 3 \end{cases}$

40. $\begin{cases} \dfrac{a}{x} + \dfrac{b}{y} = \dfrac{2}{3} \\ \dfrac{b}{x} - \dfrac{a}{y} = \dfrac{3}{4} \end{cases}$

41. $\begin{cases} \dfrac{5}{x+4} + \dfrac{6}{y-3} = 8 \\ \dfrac{9}{x+4} - \dfrac{4}{y-3} = 7 \end{cases}$

42. $\begin{cases} \dfrac{a}{x} + \dfrac{b}{y} = c \\ \dfrac{b}{x} - \dfrac{a}{y} = d \end{cases}$

43. $\begin{cases} \dfrac{x+1}{y+1} - \dfrac{x-4}{y-1} = 0 \\ \dfrac{y+4}{x-5} - \dfrac{y+2}{x-4} = 0 \end{cases}$

44. $\begin{cases} \dfrac{a}{x} - \dfrac{b}{y} = \dfrac{c}{d} \\ \dfrac{b}{x} + \dfrac{a}{y} = \dfrac{d}{c} \end{cases}$

45. $\begin{cases} \dfrac{x}{a-c} + \dfrac{y}{a-d} = 1 \\ \dfrac{x}{b-c} + \dfrac{y}{b-d} = 1 \end{cases}$

46. $\begin{cases} \dfrac{a}{x} + \dfrac{2}{y} = c \\ \dfrac{b}{x} + \dfrac{3}{y} = d \end{cases}$

47. $\begin{cases} \dfrac{x}{a+c} + \dfrac{y}{a-c} = \dfrac{1}{a-c} \\ \dfrac{x}{a+c} - \dfrac{y}{a-c} = \dfrac{1}{a+c} \end{cases}$

48. $\begin{cases} \dfrac{3}{x} + \dfrac{2}{y} = a \\ \dfrac{2}{x} - \dfrac{3}{y} = b \end{cases}$

49. $\begin{cases} \dfrac{x}{a+c} + \dfrac{y}{a-c} = \dfrac{1}{a^2-c^2} \\ \dfrac{x}{a-c} + \dfrac{y}{a+c} = \dfrac{1}{a^2-c^2} \end{cases}$

To the Teacher.—Teach 250 . . . . 252.

1. $\begin{cases} 2x+2y-3z=3 \\ 4x-3y-2z=4 \\ 3x+3y-4z=6 \end{cases}$

2. $\begin{cases} x+y-15=0 \\ y+z-17=0 \\ x+z-16=0 \end{cases}$

3. $\begin{cases} 2x+3y-6z=4 \\ 3x-3y+2z=5 \\ 4x-6y+5z=4 \end{cases}$

4. $\begin{cases} x+y-15=14 \\ x+z-14=16 \\ y+z-18=13 \end{cases}$

5. $\begin{cases} 2x+3y=2+3z \\ 3x-5y=7-2z \\ 9x-4y=4+4z \end{cases}$

6. $\begin{cases} 2x-2y-4=0 \\ 4y-2z-4=0 \\ 5x-4z-3=0 \end{cases}$

7. $\begin{cases} 3x-4y+2z=15 \\ 2x+3y-3z=15 \\ 5y-5x+4z=22 \end{cases}$

8. $\begin{cases} 3x-2y-5=12 \\ 4x-2z-7=11 \\ 7y-3z-9=17 \end{cases}$

9. $\begin{cases} 5x+3y=3z+16 \\ 5y=33+3x-2z \\ 5z-4y-2x=11 \end{cases}$

10. $\begin{cases} 3x-3y-6=12 \\ 5x-6z-9=17 \\ 2y+2z-8=30 \end{cases}$

11. $\begin{cases} 7x-5y+3z=29 \\ 6y-4z+3x=30 \\ 5z-4x+2y=37 \end{cases}$

12. $\begin{cases} 5x-3z-10=0 \\ 4y-2z-22=0 \\ 2x+3y-61=0 \end{cases}$

13. $\begin{cases} x+y+z+u=30 \\ x-y+z+u=16 \\ x+y-z+u=12 \\ x+y+z-u=18 \end{cases}$

14. $\begin{cases} 3x+2y-4z=5 \\ 4x+3z-9u=6 \\ 5x-3y-4u=6 \\ 7y+3u-6z=5 \end{cases}$

15. $\begin{cases} x+y+z-u=12 \\ x+y-z+u=18 \\ x-y+z+u=14 \\ x+y+z+u=30 \end{cases}$

16. $\begin{cases} 5x+2y-4z=29 \\ 4x-3z+3u=46 \\ 3x+5y-5u=29 \\ 4y-4u+3z=34 \end{cases}$

17. $$\begin{cases} x+y=6 \\ y+z=5 \\ z+u=4 \\ x-u=3 \end{cases}$$

18. $$\begin{cases} x+y+z=7 \\ y+z-u=3 \\ z+u+x=6 \\ u+x+y=8 \end{cases}$$

19. $$\begin{cases} x+y=a \\ y+z=b \\ x+z=c \end{cases}$$

20. $$\begin{cases} x+y+z=a \\ x+y-z=b \\ x-y+z=c \end{cases}$$

21. $$\begin{cases} \frac{1}{x}+\frac{1}{y}=7 \\ \frac{1}{y}+\frac{1}{z}=6 \\ \frac{1}{z}+\frac{1}{x}=9 \end{cases}$$

22. $$\begin{cases} \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=9 \\ \frac{1}{x}+\frac{1}{y}-\frac{1}{z}=5 \\ \frac{1}{y}+\frac{1}{z}-\frac{1}{x}=3 \end{cases}$$

23. $$\begin{cases} \frac{1}{x}+\frac{1}{y}=a \\ \frac{1}{y}+\frac{1}{z}=b \\ \frac{1}{z}+\frac{1}{x}=c \end{cases}$$

24. $$\begin{cases} \frac{a}{x}+\frac{b}{y}+\frac{c}{z}=a \\ \frac{a}{x}-\frac{b}{y}+\frac{c}{z}=b \\ \frac{b}{y}+\frac{c}{z}-\frac{a}{x}=c \end{cases}$$

25. $$\begin{cases} x+y=a \\ y+z=b \\ z+u=c \\ u-x=d \end{cases}$$

26. $$\begin{cases} x+y=a+b \\ y+z=a-c \\ z+u=a-b \\ u-x=c-a \end{cases}$$

27. $$\begin{cases} 2x+y=a \\ 2y+z=b \\ 2z+x=c \end{cases}$$

28. $$\begin{cases} 2x+y+z=a \\ 3y+z+x=b \\ 4z+x+y=c \end{cases}$$

29. $\begin{cases} \dfrac{x}{3}+\dfrac{y}{7}-\dfrac{z}{4}=2 \\[2mm] \dfrac{x}{4}+\dfrac{y}{2}-\dfrac{z}{8}=8 \\[2mm] \dfrac{x}{2}-\dfrac{z}{2}+\dfrac{y}{2}=5 \end{cases}$

30. $\begin{cases} \dfrac{x}{5}+\dfrac{y}{7}-8=0 \\[2mm] \dfrac{y}{5}-\dfrac{z}{4}-4=0 \\[2mm] \dfrac{z}{3}+\dfrac{x}{3}-9=0 \end{cases}$

31. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}-\dfrac{1}{z}=4 \\[2mm] \dfrac{1}{y}+\dfrac{1}{z}-\dfrac{1}{x}=6 \\[2mm] \dfrac{1}{z}+\dfrac{1}{x}-\dfrac{1}{y}=8 \end{cases}$

32. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}-3=5 \\[2mm] \dfrac{1}{y}+\dfrac{1}{z}-9=8 \\[2mm] \dfrac{1}{z}+\dfrac{1}{x}-6=7 \end{cases}$

33. $\begin{cases} \dfrac{x}{a}+\dfrac{y}{b}-\dfrac{z}{c}=\dfrac{1}{2} \\[2mm] \dfrac{y}{b}+\dfrac{z}{c}-\dfrac{x}{a}=\dfrac{1}{2} \\[2mm] \dfrac{z}{c}+\dfrac{x}{a}-\dfrac{y}{b}=\dfrac{1}{2} \end{cases}$

34. $\begin{cases} \dfrac{x}{a}+\dfrac{y}{b}-3=0 \\[2mm] \dfrac{y}{b}+\dfrac{z}{c}-3=0 \\[2mm] \dfrac{z}{c}+\dfrac{x}{a}-3=0 \end{cases}$

35. $\begin{cases} \dfrac{a}{x}+\dfrac{b}{y}-\dfrac{c}{z}=ac \\[2mm] \dfrac{b}{y}+\dfrac{c}{z}-\dfrac{a}{x}=ab \\[2mm] \dfrac{c}{z}+\dfrac{a}{x}-\dfrac{b}{y}=bc \end{cases}$

36. $\begin{cases} \dfrac{a}{x}+\dfrac{b}{y}=a+4 \\[2mm] \dfrac{b}{y}+\dfrac{c}{z}=b+3 \\[2mm] \dfrac{c}{z}+\dfrac{a}{x}=a+2 \end{cases}$

37. $\begin{cases} \frac{1}{2}x+\frac{1}{6}y-\frac{1}{3}z=6 \\ \frac{1}{3}y+\frac{1}{4}z-\frac{1}{3}x=8 \\ \frac{1}{3}z+\frac{1}{6}x-\frac{1}{9}y=8 \end{cases}$

38. $\begin{cases} \frac{8}{9}x-\frac{3}{7}y-5=9 \\ \frac{5}{6}y-\frac{4}{9}z-8=7 \\ \frac{3}{5}z-\frac{1}{3}x-7=8 \end{cases}$

To THE TEACHER. — Show how problems involving simultaneous equations are solved. See 253 and examples following.

PROBLEMS INVOLVING SIMULTANEOUS EQUATIONS.

1. The sum of two numbers is sixty-eight, and their difference is thirty-four. Find the larger number.

2. The sum of two numbers is 72, and their difference exceeds $\frac{1}{3}$ of the smaller by nine. Find the numbers.

3. The sum of two numbers is 136, and two-thirds of the larger is equal to three-fourths of the smaller. How much does the larger number exceed the smaller?

4. The sum of the two digits of a number is eleven; and if twenty-seven be added to the number, the digits will be interchanged. Find the number.

5. If the greater of two numbers is divided by the less, the quotient is 5 and the remainder 2; but if 6 times the smaller is divided by the greater, the quotient is 1 and the remainder 7. Find the numbers.

6. If 3 be added to the numerator of a certain fraction, its value will be $\frac{1}{2}$; and if 1 be subtracted from the denominator, its value will be $\frac{1}{3}$. Find the fraction.

7. A farmer has two hundred twenty sheep in two fields. If thirty-five are taken out of the first field and put into the second, there will be the same number in each field. How many are there in each field?

8. Five times the greater of two numbers exceeds $\frac{1}{5}$ of the less by 97, and five times the less exceeds $\frac{1}{5}$ of the greater by 71. Find the sum of the numbers.

9. If eight be added to the sum of the two digits of a number, the result will be four times the left-hand digit; and if eighteen be added to the number, the digits will be interchanged. Find the number.

10. In an election one thousand four hundred ninety men voted for two candidates, and the candidate elected had a majority of two hundred forty-six. How many votes did each candidate receive?

11. A man sold 70 sheep for $240. He sold some of them at $3 a head and the rest at $4 a head. How many sheep did he sell at four dollars a head?

12. The sum of the ages of A and B is ninety-six years. If B were twice as old, his age would exceed A's age by eighteen years. Find the age of each.

13. If the sum of two numbers be divided by 6, the quotient will be 18 and the remainder 2; and if their difference be divided by 5, the quotient will be 15 and the remainder 3. Find the numbers.

14. The sum of the three digits of a number is 12. The digit in tens' place is $\frac{1}{2}$ the sum of the other two digits; and if 198 be added to the number, the first and last digits will be interchanged. Find the number.

15. Eight men and six boys earn seventeen dollars a day, and at the same wages, nine men and five boys earn $18.25. How much does each man earn per day?

16. When a man was married, his age was $\frac{5}{6}$ of his wife's age; eight years afterward his age was $\frac{6}{7}$ of his wife's age. How old were they when they were married?

17. If a rectangular plot of land were 10 feet longer and 20 feet wider, the area would be 3400 square feet more than it is; but if the length were 10 feet more and the width 20 feet less, the area would be 1800 square feet less than it is. Find the area of the plot.

18. If 2 be added to both terms of a certain fraction, its value will be $\frac{3}{4}$; and if 4 be subtracted from both terms, its value will be $\frac{1}{2}$. Find the fraction.

19. The sum of the digits of a number of two figures is 13; and if 45 be subtracted from the number, the digits will be interchanged. Find the number.

20. At an election A's majority was three hundred eighty-four, which was three-elevenths of the whole number of votes. How many votes did each receive?

21. A dealer bought 40 barrels of apples and 20 barrels of pears for \$105. He sold the apples at a profit of 30%, and the pears at a profit of 20%, receiving for all \$131. How much did he receive for the apples?

22. A and B can do a piece of work in 4 days; B and C can do it in 5 days; A and C can do it in $6\frac{2}{3}$ days. In how many days can each do the work?

23. Seven years ago B was three times as old as A, but in five years he will be only twice as old. Required the age of each seven years ago.

24. The sum of the two digits of a number is 12; and if the number be divided by the sum of its digits, the quotient will be seven. Find the number.

25. A, B, and C have \$1380. If B gives A \$70, A will have \$80 more than B; but if B gets \$60 from C, B and C will have the same sum. How much has each?

26. In three years a certain sum of money, at simple interest, will amount to \$966, and in five years it will amount to \$1050. Find the sum invested.

27. The average age of three persons is 60 years. The average age of the first and second is 52 years, and of the second and third is 70 years. Find the age of each.

28. A miller mixes corn worth \$.60 a bushel with oats worth \$.45 a bushel, making a mixture of 100 bushels, worth \$.54 a bushel. How many bushels of corn does he use?

29. Two men are 12 miles apart. If they set out at the same time and travel in the same direction, they will be together in six hours; but if they travel in opposite directions toward each other, they will be together in two hours. At what rate do they travel?

30. A man can row 15 miles down a river in 3 hours, but it takes 5 times as long to row the same distance up the river. At what rate can he row on still water?

31. The sum of the first and last digits of a number exceeds six times the tens' digit by three. If the first and last digits be interchanged, the number will be diminished by 99; and if the digits in units' and tens' places be interchanged, the number will be increased by 45. Find the number.

32. A man paid $11 for oranges, buying some of them at eighteen for 25 cents, and the rest at sixteen for 25 cents. He sold them all at 25 cents a dozen and gained $4.50. How many oranges did he buy?

33. If the numerator of a fraction be doubled and 2 added to the denominator, its value will be $\frac{3}{5}$; and if the denominator be doubled and 3 added to the numerator, its value will be $\frac{3}{8}$. What is the fraction?

34. A and B together had $99,000. After A had lost $\frac{2}{5}$ of his money in speculation and B $\frac{3}{8}$ of his, they together had $61,000 left. How much had each at first?

35. A man invested $20,000, partly in 4% bonds, and partly in 5% bonds. The annual income from the 4% bonds exceeds the annual income from the 5% bonds by $125. How much did he invest at 5 per cent?

36. A and B can do a piece of work in a days; B and C can do it in b days; A and C can do it in c days. In how many days can each do the work?

37. Eight years ago a man was three times as old as his son, and five years hence four times the son's age will equal twice the father's age. Find the age of each.

38. A certain number exceeds five times the sum of its two digits by twenty; and if the number be divided by the left-hand digit, the quotient will be ten and the remainder five. Find the number.

39. After A had sold sixty-two sheep to B and eighteen to C, they each had the same number. Before A made these sales, he lacked only eight sheep of having as many as B and C together. How many have all?

40. If a rectangular piece of paper were three inches shorter and two inches wider, the area would be 11 square inches less; but if a strip two inches wide be cut off around the paper, the area will be diminished 124 square inches. Find the area of the paper.

41. A miller bought 40 bushels of corn and 30 bushels of oats for $45. At another time he bought at the same prices 43 bushels of oats and 55 bushels of corn for $62.75. Find the cost of four bushels of each.

42. A crew can row 12 miles down a river in 2 hours, but it takes them twice as long to row the same distance up the river. Find the rate of the current.

43. In nine months a certain sum of money, at simple interest, will amount to $996, and in sixteen months it will amount to $1024. Find the sum invested.

44. In 7 hours A walks 11 miles more than B does in 6 hours; and in 12 hours B walks 23 miles more than A does in 5 hours. How many miles does each walk per hour?

45. The sum of the two digits of a number is 15; and if the digits be interchanged, the resulting number will exceed the given number by 27. Find the number.

46. A man has silver dollars, half-dollars, and quarters, together worth \$46. One-half of his dollars and one-fifth of his half-dollars are worth \$12.50; one-seventh of his half-dollars and one-third of his quarters are worth six dollars. How many coins has he?

47. A and B can do a piece of work in $8\frac{4}{7}$ days; but if they work together six days, B can finish the work in six days more. In how many days can each do the work?

48. Two wheelmen race a mile, and A wins by 5 seconds. In the second trial B has a start of 56 yards and wins by 2 seconds. Assuming that each rides at a uniform rate, in how many seconds can each ride a mile?

49. A boy bought some peaches at the rate of three for five cents, and some at a cent apiece, paying \$3 for all of them. He sold them all at two cents apiece and gained \$1.40. How many peaches did he buy?

50. The numerator of the larger of two fractions is 5, and the numerator of the smaller fraction is 7. The sum of the fractions is $1\frac{5}{12}$; and if the numerators be interchanged, their sum will be $1\frac{7}{12}$. Find the fractions.

51. A man bought a piece of land. If he had bought 20 acres more for the same money, it would have cost \$5 less per acre; but if he had bought ten acres less for the same money, it would have cost four dollars more per acre. How much did he pay for the land?

52. In m years a certain sum of money, at simple interest, will amount to a dollars, and in n years it will amount to b dollars. Find the sum invested.

53. A merchant sold thirty-six yards of silk, part of it at \$1.25 a yard, and the rest at two dollars a yard, receiving fifty-seven dollars for the whole. How many yards of the better silk did he sell?

54. There are three numbers such that two-thirds of the first and three-fifths of the second together make eighty-five; two-thirds of the second and three-fifths of the third together make one hundred four; two-thirds of the third and three-fifths of the first together make ninety-six. Find the sum of the three numbers.

55. Two men race 440 yards. In the first trial A gives B a start of 13 seconds, and B wins by 20 yards; in the second trial A gives B a start of 88 yards, and the race is a tie. Assuming that each runs at a uniform rate, how many yards can each run in a minute?

56. The sum of the three digits of a number is 14. If the number, diminished by 10, be divided by the sum of its digits, the quotient will exceed 5 times the sum of the first and last digits by 8; and if 197 be subtracted from the number, all the digits will be the same as the tens' digit in the given number. Find the number.

57. A grocer has two measures. Twelve times the capacity of the larger or eighteen times that of the smaller will fill a certain vessel. If he fills the vessel, using both measures 16 times in all, how many times must he use each?

58. A man can row 3 miles with the current in 45 minutes, and 2 miles against the current in 1 hour and 20 minutes. At what rate can he row on still water?

59. If the length of a rectangle were five feet less and the width four feet more, the figure would be a square of the same area as the given rectangle. What is the area of the rectangle?

60. A tank has three pipes. A and B will empty it in 1 hour; B and C will empty it in 2 hours; A and C will empty it in 1 hour and 12 minutes. How many minutes will it take each pipe to empty the tank?

61. A pound of tea and four pounds of coffee cost \$2. At prices 30% higher, 3 pounds of tea and 6 pounds of coffee would cost \$5.46. Find the price of the tea.

62. A marketman bought eggs, some at 3 for 4 cents, and some at the rate of 5 cents for 4, paying \$4 for all of them. He found 22 that were unsalable, and he sold the remainder at 22 cents a dozen, making a profit of one dollar twenty-eight cents. How many did he buy?

63. A certain number is expressed by two figures. If the number be divided by the sum of its digits, the quotient will be seven and the remainder six. If the digits be interchanged and the resulting number divided by the sum of its digits, the quotient will be three and the remainder two. Find the number.

64. A man bought a certain number of sheep. If he had bought 5 more for the same money, they would have cost 50 cents less apiece; but if he had bought 5 less for the same money, they would have cost 75 cents more apiece. How much did he pay for the sheep?

65. A wife, son, and daughter inherit \$30,000. One-fifth of the wife's share added to one-third of the son's share and one-fourth of the daughter's share makes \$7500; and $\frac{1}{3}$ of the wife's share added to $\frac{1}{5}$ of the daughter's share exceeds $\frac{1}{2}$ of the son's share by \$1700. What part of the whole sum does the daughter receive?

66. If I invest my money at 5 per cent for a given time, I shall receive \$405 interest; but if I invest it at the same rate for $2\frac{1}{2}$ years longer, I shall receive \$1080 interest. How much money have I?

67. A and B can do a piece of work in a days; but if they work together b days, B can finish the work in c days more. In how many days can each do the work?

68. A boy bought at one time 4 apples and 5 pears for 16 cents; at another time, 3 pears and 6 peaches for 21 cents; at another time, 5 pears and 6 oranges for 28 cents; and at another time, 4 peaches and 5 oranges for 25 cents. Find the price of each kind of fruit.

69. Three boys played at marbles. B and C each won from A as many as they had at first; A and C each won from B as many as they then had; and A and B each won from C as many as they then had. Each boy then had 88 marbles. How many did A and B together have at first?

70. A boatman can row a miles down a river in m minutes and b miles up the river in n minutes. At what rate per hour can he row on still water?

71. A and B engaged in business, B having $2400 more capital than A. After A had lost one-fifth of his capital and B one-third of his, they found that they each had the same sum left. How much did both invest?

72. A merchant has tea worth 50 cents a pound, and another kind worth one dollar a pound. How many pounds of each kind must he take to make a mixture of twenty-five pounds, worth seventy-two cents a pound?

73. If a certain floor were 3 feet longer and one foot wider, it would require 7 square yards more of carpet to cover it; but if the length were 3 feet less and the width 2 feet less, it would require 8 square yards less of carpet to cover it. Find the area of the floor.

74. The hundreds' digit in a number exceeds the tens' digit by four times as much as the units' digit exceeds the tens'. If the number be divided by the sum of its digits, the quotient will be 70 and the remainder 3; and if 594 be subtracted from the number, the first and last digits will be interchanged. Find the number.

75. Half of A's money is \$1400 more than one-fifth of B's and C's together; five-ninths of B's is equal to the excess of A's over C's; and the sum of A's and C's lacks \$4000 of being 3 times B's. How much have all?

76. A merchant divided a sum of money among his four employees, giving A \$35 more than D. A received half as much as the other three; B received one-third as much as the other three; and C received one-fourth as much as the other three. How much did all receive?

77. In m months a certain sum of money, at simple interest, will amount to a dollars, and in n months it will amount to b dollars. Find the sum invested.

78. A boy bought a certain number of oranges. If he had bought 20 more for the same money, they would have cost half a cent apiece less; but if he had bought 20 less for the same money, they would have cost a cent apiece more. How much did he pay for the oranges?

79. A train, after running 1 hour from A toward B, met with an accident, which delayed it 30 minutes. The train then proceeded at $\frac{3}{5}$ of its former rate and arrived at B 90 minutes late. If the accident had happened 15 miles nearer A, the train would have been 100 minutes late. How far from B did the accident occur?

80. A boy expended one dollar for pears, buying some at two for a cent, and the rest at three for a cent. He sold them all at the rate of two cents for three and made fifty cents. How many pears did he buy?

81. A boatman can row down a river a distance of 15 miles, and back again, in 9 hours; and the current is such that he can row $3\frac{1}{2}$ miles up the stream in the same time that he can row 7 miles down the stream. How long does it take him to go down the fifteen miles?

82. A and B can do a piece of work in $6\frac{2}{3}$ days; B and C can do it in $8\frac{2}{11}$ days. If A can do $1\frac{1}{2}$ times as much as C, in how many days can each do the work?

83. A man has \$ 27,000 invested, partly at $3\frac{1}{2}$ per cent, partly at 4 per cent, and partly at 5 per cent, from which he receives an annual income of \$ 1126. The sum invested at $3\frac{1}{2}\%$ is equal to $\frac{4}{5}$ of the sum of the other two investments. Find the sum invested at $3\frac{1}{2}\%$.

84. Two athletes run half a mile. In the first trial A gives B a start of 88 yards and is beaten by 8 seconds; in the second trial B has a start of 36 seconds and is beaten by $7\frac{1}{3}$ yards. Assuming that each runs at a uniform rate, in how many minutes can each run a mile?

85. A certain number is expressed by two digits. If the number be divided by ten, the quotient will be seven and the remainder five; but if the number be divided by the sum of its digits, the quotient will be six and the remainder three. Find the number.

86. A, B, and C have farms adjoining. If A should buy 40 acres of B, A would have twice as many acres as B would have left; if B should buy 60 acres of C, C would have left $\frac{2}{3}$ as many acres as B would then have; if C should buy 125 acres of A, C would have 3 times as many acres as A would have left. How many acres have all?

87. A miller has rye worth 70 cents a bushel, corn worth 60 cents a bushel, and oats worth 40 cents a bushel. He wishes to make a mixture of 100 bushels of the three kinds of grain, worth 56 cents a bushel, and use twenty bushels of rye. How many bushels of corn must he use?

88. A dealer sold 50 barrels of flour at an average price of \$ 3.22, selling part of it at \$ 3 and the rest at \$ 3.50. How many barrels did he sell at \$ 3.50?

89. The difference between two fractions, which have the same denominator, is $\frac{2}{3}$. If 2 be added to the numerator of the smaller, its value will be $\frac{1}{2}$ of the larger; if 2 be subtracted from the numerator of the larger, its value will be 3 times the smaller. Find the fractions.

90. A grocer has three kinds of tea. A mixture of 4 pounds of the first and 6 pounds of the second is worth \$.52 a pound; a mixture of 6 pounds of the second and 4 pounds of the third is worth \$.68 a pound; a mixture of 5 pounds of the first and 5 pounds of the third is worth \$.60 a pound. Find the price of each kind of tea.

91. A railway train, after running 45 minutes from A toward B, met with an accident, which delayed it 20 minutes. The train then proceeded at $\frac{3}{4}$ of its former rate and arrived at B 65 minutes late. If the accident had happened 36 miles farther on, the train would have been only 50 minutes late. Find the distance from A to B.

92. A man made two investments, one at $3\frac{1}{2}$ per cent, and the other at $4\frac{1}{2}$ per cent. His income from both investments in two years was \$2600. If he had made the first investment at $4\frac{1}{2}$ per cent, and the second at $3\frac{1}{2}$ per cent, the income from both investments would have been \$2520. Find the whole sum invested.

93. A man bought 600 oranges, some at four for 10 cents, and the rest at six for 10 cents. He lost two dozen and sold the remainder at 35 cents a dozen, gaining \$3.80. How many of the better ones did he buy?

94. Two wheelmen race a mile. In the first trial A gives B a start of 176 yards and wins by 30 seconds; in the second trial A gives B a start of 1 minute and 9 seconds, and B wins by 66 yards. Assuming that each rides at a uniform rate, in what time can each ride a mile?

95. A cistern can be emptied by two pipes in 3 hours. If both pipes are open 2 hours, and then A is closed, it will take B 2 hours and 30 minutes longer to empty it. How long will it take each pipe to empty it?

96. The sum of the three digits of a number is ten, the tens' digit being zero. If the first and last digits be interchanged, the number will be increased by five hundred ninety-four; and if the number be divided by five times the sum of its digits, the quotient will be four and the remainder eight. Find the number.

97. A boatman rowed down a river, which flows at the rate of $1\frac{3}{8}$ miles an hour, going a certain distance in 1 hour and 12 minutes. It took him 3 hours and 36 minutes to return. How many miles did he row?

98. A man divided a sum of money equally among a certain number of boys. If there had been a more boys to receive the same sum, each one would have received m dollars less; but if there had been b less boys to receive the same sum, each one would have received n dollars more. Find the number of boys and each boy's share.

99. A, B, and C together had one hundred twenty dollars. A gave to B and C as much as each of them already had; B gave to A and C as much as each of them then had; C gave to A and B as much as each of them then had, and it was then found that they had equal amounts. How much did B and C together have at first?

100. A boy has fifteen dollars in 5-dollar bills, half-dollars, quarters, and dimes, and he has eight times as many coins as bills. If his half-dollars were quarters and his quarters half-dollars, he would have $14.25; if his dimes were quarters and his quarters dimes, he would have $15.15. How many pieces of money has he?

TO THE TEACHER. — Teach 254 . . . . 263.

DEFINITIONS,
PRINCIPLES AND RULES.

DEFINITIONS, PRINCIPLES, AND RULES.

1. Quantity is the amount or extent of anything that can be measured.

2. The Measure of a quantity is the number of times it contains some other quantity of the same kind taken as a *unit*, or *standard of comparison.*

3. Number is the relation between any quantity and its measuring unit. All quantities can therefore be expressed by numbers. Number and quantity are synonymous terms, but the term *quantity* is used in algebra more frequently than the term *number*. In this treatise both terms will be found.

4. Mathematics is the science which treats of the properties, measurement, and relation of quantities. Prominent among the branches of mathematics are Arithmetic, Algebra, and Geometry.

5. Arithmetic is that branch of mathematics which treats of the nature and properties of numbers, the various methods of combining and resolving them, and their application to practical affairs.

6. Algebra is that branch of mathematics in which figures and letters are used to represent numbers. It treats of numbers in the way of finding general truths in regard to them, but the special function of algebra is to treat of the nature and transformation of equations, the methods of

solving them, and their application to the solution of problems.

NOTATION.

7. **A System of Notation** is a system of symbols by means of which quantities, the relations between them, and the operations to be performed upon them, can be more concisely represented than by the use of words.

8. **A Mathematical Symbol** is a mark or character which has a distinct meaning.

9. The symbols employed in algebra are as follows:

1. Symbols of Number.
2. Symbols of Operation.
3. Symbols of Relation.
4. Symbols of Grouping.
5. Symbols of Continuation.
6. Symbols of Deduction.

SYMBOLS OF NUMBER.

10. **The Symbols of Number** are the Arabic figures, and letters of the alphabet.

In algebra, letters are used to represent numbers, and these, unlike the figures used in arithmetic, represent any numbers that may be assigned to them. Letters are thus used as general symbols of numbers. It must be remembered, however, that in any particular example or problem, a given letter represents the same number or numbers throughout the whole solution.

TO THE TEACHER.—Impress upon students from the beginning, and always emphasize the fact, that the letters used in algebra are not arbitrary symbols without meaning or signification, but that every letter, every combination of letters, and every combination of figures and letters, represents some number.

11. When a problem in algebra involves several *like* numbers which differ in numerical value, they are often represented by the same letter with different *accents* or different *subscript figures.*

a' is read *a prime,*	a'' is read *a second,*
a''' is read *a third,*	a'''' is read *a fourth,*
a_1 is read *a sub one,*	a_2 is read *a sub two,*
a_3 is read *a sub three,*	a_4 is read *a sub four.*

12. The Algebraic Notation is the method of expressing numbers by letters and figures. Since letters are used, the algebraic notation is called a *literal notation.*

13. It will be remembered that the whole of a number expressed in the decimal notation is equal to the *sum* of the parts represented by the several digits.

$$456 = 400 + 50 + 6.$$
$$2178 = 2000 + 100 + 70 + 8.$$

This is not the law of the algebraic notation. When a number is expressed by several letters, written without any sign between them, the whole number is equal to the *product* of the parts represented by the several letters. If figures are used with the letters to represent the number, the sign of multiplication is also understood between the figures and letters.

$$abxy = a \times b \times x \times y.$$
$$6\,abc = 6 \times a \times b \times c.$$
$$25\,bxy = 25 \times b \times x \times y.$$

This is known as the *law of the algebraic notation.*

14. LAW OF THE ALGEBRAIC NOTATION. — ***When several letters are written in connection without any sign between them, their product is indicated.***

15. Known Numbers, or Known Quantities, are those whose values are given. They are represented by figures, or by the first letters of the alphabet.

16. Unknown Numbers, or Unknown Quantities, are those whose values are to be found. They are represented by the last letters of the alphabet.

Symbols of Operation.

17. The Symbols of Operation, generally called *signs,* are characters that indicate the operations to be performed with numbers. They are as follows:

1. The Sign of Addition.
2. The Sign of Subtraction.
3. The Sign of Multiplication.
4. The Sign of Division.
5. The Sign of Ratio.
6. The Sign of Involution.
7. The Sign of Evolution.

18. The Sign of Addition is an upright cross. $+$ It is read *plus,* and it indicates that the number after the sign is to be added to the number before it.

19. The Sign of Subtraction is a short horizontal line. $-$ It is read *minus,* and it indicates that the number after the sign is to be subtracted from the number before it.

20. The Sign of Multiplication is an oblique cross. $\times$ It is read *multiplied by,* or *times,* and it indicates that the number before the sign is to be multiplied by the number after it.

Multiplication is also indicated in algebra by writing the letters in connection without any sign between them; also by placing a dot between the letters.

$$6 \times a \times b \times c = 6abc = 6 \cdot a \cdot b \cdot c.$$

21. **The Sign of Division** is a short horizontal line placed between two dots. $\div$ It is read *divided by,* and it indicates that the number before the sign is to be divided by the number after it.

Division is also indicated by writing the dividend over the divisor with a line between them; also by writing the divisor at the right of the dividend with the colon between them.

$$a \div b = \frac{a}{b} = a : b.$$

22. **The Sign of Ratio** is two dots in the form of a colon. : It is read *is to,* and it means the same as division.

23. **An Exponent** is a symbol of number written at the right and a little above another symbol of number. It may be an integer or a fraction. When no exponent is expressed, 1 is always understood.

$$a^3 \qquad x^{\frac{1}{3}} \qquad c^n \qquad b^m \qquad y^{\frac{1}{n}}$$

24. **A Positive Integral Exponent,** or **Symbol of Involution,** indicates that the quantity affected by it is to be taken as a factor as many times as there are units in the exponent.

$$a^3 = a \times a \times a. \qquad m^5 = m \times m \times m \times m \times m.$$

25. **A Power** of a quantity is the product obtained by using the quantity a certain number of times as a factor.

27 is the third power of 3. a^3 is the third power of a.

32 is the fifth power of 2. x^4 is the fourth power of x.

26. **The Factors** of a quantity are the quantities which multiplied together will produce that quantity.

The factors of a number or quantity may be equal or unequal. One of the equal factors of a number is a *root.*

27. A Root of a number is one of its equal factors.

Two is a root of 4, 8, 16, and 32, because it is one of the equal factors of each of them; 3 is a root of 9, 27, and 81, because it is one of the equal factors of each of them.

28. The Square Root of a number or quantity is one of its two equal factors.

2 is the square root of 4.
3 is the square root of 9.
5 is the square root of 25.
7 is the square root of 49.

29. The Cube Root of a number or quantity is one of its three equal factors.

3 is the cube root of 27.
4 is the cube root of 64.
5 is the cube root of 125.

30. The Sign of Evolution is the *radical sign.* $\sqrt{}$ When it is placed before a number, it indicates that a root of the number is required. Evolution is also indicated by a fractional exponent.

The radical sign alone indicates the square root.

Any other root than the square root is indicated by a number, called the *index,* written in the opening of the radical sign.

The expression $\sqrt{36}$ means the square root of 36.

The expression $\sqrt[4]{16a}$ means the fourth root of $16a$.

Symbols of Relation.

31. The Symbols of Relation are the following:

1. The Sign of Equality.
2. The Signs of Inequality.
3. The Sign of Proportion.

32. The Sign of Equality is two short horizontal lines placed one above the other. $=$ It is read *equals,* or *is equal to,* and it indicates that the numbers between which it is placed are equal.

33. The Signs of Inequality are $>$ and $<$. They are read *is greater than,* and *is less than,* the opening of the sign being toward the greater number.

34. The Sign of Proportion is the double colon. :: It is read *as,* and its signification is the same as the sign of equality.

Symbols of Grouping.

35. The Symbols of Grouping are the parenthesis ( ), brace { }, brackets [ ], and vinculum $\overline{\quad\quad}$. They indicate that the quantities inclosed by them are to be treated as one quantity.

The vinculum is used in connection with the radical sign. The line between the numerator and denominator of a fraction also has the force of a vinculum. These symbols of grouping are also called *symbols of aggregation.*

36. The Sign of Continuation is a series of dots . . . . or dashes – – – –. It is read *and so on.*

37. The Sign of Deduction is $\therefore$. It is read *hence* or *therefore.*

Algebraic Expressions.

38. An Algebraic Expression is the representation of any quantity in the algebraic notation.

xy $\qquad$ $2a-3b$ $\qquad$ $5a^2b^3$

39. A Term is an algebraic expression whose parts are not connected by the sign of addition or subtraction.

abc $\qquad$ $2a\times 3x$ $\qquad$ $8a\div 2c$ $\qquad$ $2a(ab-3c)$

Note.—It must be remembered that when no sign of grouping is used, the signs of multiplication and division indicate operations *within* a term, while the signs of addition and subtraction indicate operations *between* terms.

When a sign of grouping is used, the sign of addition or subtraction may indicate an operation *within* a term.

$2a(x - y)$ and $3x(2a + b - 3c)$ are terms.

40. Positive Terms are terms preceded by the sign of addition. If no sign is expressed, the term is positive.

$3ac \qquad +xy \qquad +4mn \qquad 8abx$

41. Negative Terms are terms preceded by the sign of subtraction.

$-abc \qquad -6ax \qquad -xyz \qquad -9ac$

42. The signs of addition and subtraction, besides being used as symbols of operation, are also used in algebra to denote qualities that are opposite in kind, direction, or effect.

If a man gains $500 and then loses $200, he has a positive gain of $300. If a man gains $300 and then loses $700, he has a negative gain, or loss, of $400. If a man walks east from a certain point 9 miles and then walks west 6 miles, he has made a positive advance of 3 miles in the direction in which he started. If he walks east 10 miles and then walks west 14 miles, he has made a negative advance of 4 miles, or he is 4 miles *west* of the starting point. If a man's assets are $8000 and he owes $3500, he is really worth only $4500. If his assets are $9000 and he owes $15,000, his net indebtedness is $6000.

In all the preceding examples, the quantities are so related that when they are combined, the numerically smaller number cancels part of the larger number. To indicate the opposite character of these quantities, the terms *positive* and *negative* are used.

43. A Coefficient is a factor of a term.

By the law of the algebraic notation, 5, a, b, and c are the factors of the term $5\,abc$. The numerical factor is 5, and the literal factors are a, b, and c. Every algebraic term has a numerical factor. When no numerical factor is expressed, 1 is always understood.

The first factor of a term, numerical or literal, is generally thought of as the coefficient, but any factor of a term, or the product of any factors, may be considered the coefficient of the rest of the term.

In the term $4\,xy$, 4 is the coefficient of xy; $4\,x$ is the coefficient of y; $4\,y$ is the coefficient of x.

In the term $abcx$, 1 is the coefficient of $abcx$; a is the coefficient of bcx; ab is the coefficient of cx; abc is the coefficient of x.

In the term $4(a-c)$, 4 is the coefficient of $(a-c)$. In the term $(a+b)x$, $(a+b)$ is the coefficient of x.

44. Similar Terms are terms having the same letters and the corresponding letters affected with the same exponent.

$$ab^2c^3 \qquad 5\,ab^2c^3 \qquad 8\,ab^2c^3 \qquad 7\,ab^2c^3$$

NOTE. — The following terms are partly similar:

abx, bcx, and $2\,dx$ are similar with reference to x.
axy, bxy, and $4\,xy$ are similar with reference to xy.

45. Dissimilar Terms are terms containing different letters, or terms containing the same letters, the corresponding letters being affected with different exponents.

ab, cd, and xy are dissimilar terms.
x^2y, xy^2, and x^3y^3 are dissimilar terms.

46. The Degree of a term is the number of literal factors it contains. It is determined by the sum of the exponents.

$4\,ac$, ab, and a^2 are terms of the second degree.
$5\,abc$, xy^2, and z^3 are terms of the third degree.
$6\,a^2b^2$, xyz^2, and $6\,ac^3$ are terms of the fourth degree.

47. **A Monomial** is an algebraic expression of one term.

$3abx$ $\quad 4a^2c$ $\quad x^2y^4z^3$ $\quad 16ax^m$

NOTE.—A monomial, or algebraic expression of one term, is also called a *simple expression.*

48. **A Polynomial** is an algebraic expression of two or more terms.

$2a-3x$ $\quad a+b-3$ $\quad 2a-bx+3cd-4ay$

NOTE.—A polynomial, or algebraic expression of two or more terms, is also called a *compound expression.*

49. **A Binomial** is a polynomial of two terms.

$a+b$ $\quad 5x-1$ $\quad 3ac+5xy$

50. **A Trinomial** is a polynomial of three terms.

$a+b-c$ $\quad a-x-1$ $\quad 2a^2b-4b^3c+x^2y^3z^4$

51. **Homogeneous Terms** are terms of the same degree.

a^2c^3 $\quad 7x^4y$ $\quad 2acx^3$ $\quad abcdx$

52. **A Homogeneous Polynomial, or Homogeneous Quantity,** is a polynomial whose terms are of the same degree.

$$3a^3c+x^2yz-8bc^3+7x^4-2bcd^2$$

53. **An Arranged Polynomial** is a polynomial whose terms are arranged according to the exponents of the same letter, the exponents ascending or descending in regular order.

$$3a^5b+2a^4c-5a^3d+6a^2-a+2$$

54. **The Reciprocal** of a quantity is 1 divided by that quantity.

55. **The Numerical Value** of an algebraic expression is the number obtained by assigning a numerical value to each letter and performing the indicated operations.

ADDITION.

56. Addition is the process of uniting two or more quantities into the simplest expression called their sum.

57. To add two or more similar positive terms.

Since $5a$ is 5 times a, and $4a$ is 4 times a, the sum of $5a$ and $4a$ is 5 times a plus 4 times a, or 9 times a, or $9a$; for 5 times any quantity plus 4 times the same quantity is 9 times that quantity.

If more than two positive terms are to be added, the reasoning is the same.

58. To add two or more similar negative terms.

Since $-7a$ is 7 times $-a$, and $-5a$ is 5 times $-a$, the sum of $-7a$ and $-5a$ is 7 times $-a$ plus 5 times $-a$, or 12 times $-a$, or $-12a$; for 7 times any quantity plus 5 times the same quantity is 12 times that quantity.

If more than two negative terms are to be added, the reasoning is the same.

59. To add similar positive and negative terms.

$$\begin{array}{r} 4a \\ -2a \\ 7a \\ -6a \\ \hline 3a \end{array}$$

The sum of the positive terms is $11a$, and the sum of the negative terms is $-8a$. When these terms are combined, the $-8a$ cancels $8a$ of the $11a$, and the sum is $3a$.

$$\begin{array}{r} 5a \\ -8a \\ 3a \\ -9a \\ \hline -9a \end{array}$$

The sum of the positive terms is $8a$, and the sum of the negative terms is $-17a$. When these terms are combined, the $8a$ cancels $-8a$ of the $-17a$, and the sum is $-9a$.

60. PRINCIPLES.—1. *Only similar terms, or terms partly similar, can be united into one.*

2. *The sum of two quantities, one positive and the other negative, is their numerical difference with the sign of the greater prefixed.*

61. To add similar terms.

RULE.—*If the signs are alike, add the coefficients, to their sum annex the literal part, and give the result the common sign.*

If the signs are unlike, add the positive terms and the negative terms separately, subtract the less sum from the greater, and give the result the sign of the greater.

EXAMPLES.

$4ab$	$-3bc$	$5xy$	$-2a^2x$	$7x^3y$
$2ab$	$-5bc$	$-2xy$	$7a^2x$	$-5x^3y$
ab	$-\ bc$	xy	$-\ a^2x$	$2x^3y$
$6ab$	$-3bc$	$8xy$	$-9a^2x$	x^3y
$3ab$	$-\ bc$	$-2xy$	$3a^2x$	$-2x^3y$
$16ab$	$-13bc$	$10xy$	$-2a^2x$	$3x^3y$

62. To add dissimilar terms.

RULE. — *Write the terms in succession, giving each term its own sign.*

EXAMPLES.

$6ac$	$5xy$	$8cd$
$2bd$	$-2cd$	$3ax$
$-c$	z	$-2y$
$6ac+2bd-c$	$5xy-2cd+z$	$8cd+3ax-2y$

63. To add terms that are partly similar.

RULE. — *Indicate the addition of the dissimilar parts, and write the result in a parenthesis as a compound coefficient of the similar part.*

EXAMPLES.

ax	ac	$3y$	ax^2
bx	$-bc$	ay	$-bx^2$
$-x$	$2c$	$-2y$	ax^2
$(a+b-1)x$	$(a-b+2)c$	$(a+1)y$	$(2a-b)x^2$

64. To add polynomials.

RULE. — *Write the polynomials so that similar terms shall be in a column, and add each column according to the rule for addition of similar terms.*

EXAMPLE.

$$\begin{array}{rrrrrrr}
5ab & +3ac & -2bc & +3bd & +5xy & -7z^2 & \\
3ab & -ac & & -7bd & +2xy & & \\
 & 5ac & -bc & & & +3z^2 & \\
-ab & & +3bc & & -4xy & & +6 \\
 & ac & & +2bd & & +4z^2 & \\
\hline
7ab & +8ac & & -2bd & +3xy & & +6
\end{array}$$

SUBTRACTION.

65. Subtraction is the process of finding the difference between two quantities.

66. The Minuend is the quantity from which another quantity is to be subtracted.

67. The Subtrahend is the quantity to be subtracted.

68. The Remainder is the quantity obtained by subtraction.

69. PRINCIPLE. — *The minuend is equal to the sum of the subtrahend and remainder.*

70. To perform subtraction.

$$\begin{array}{rrrrrrrr}
9a & 3a & -9a & -3a & -9a & -3a & 9a & 3a \\
3a & 9a & -3a & -9a & 3a & 9a & -3a & -9a \\
\hline
6a & -6a & -6a & 6a & -12a & -12a & 12a & 12a
\end{array}$$

The above examples represent all possible cases in subtraction with reference to the signs and relative numerical value of minuend and subtrahend. The signs may be alike or unlike, and the minuend may be numerically greater or less than the subtrahend.

Since algebra deals with positive and negative quantities, and since the subtrahend may be either numerically less or *greater* than the minuend, it is evident that the method of subtraction pursued in arithmetic cannot, in every case, under these different conditions, give the correct result.

There is nothing in the nature of subtraction in algebra that makes it necessary to proceed in the same manner in every example. It is only necessary to obtain a quantity which added to the subtrahend will produce the minuend.

It is easily determined by inspection of the above examples that $6a$, $-6a$, $-6a$, $6a$, $-12a$, $-12a$, $12a$, and $12a$ are the quantities which added to the several subtrahends will produce the minuends.

An examination of the work shows that each remainder could have been obtained by changing the sign of the subtrahend and adding it to the minuend.

Since the correct remainder, *in every example*, may be obtained in this manner, and since it is desirable to have a uniform method for all examples, this method of subtraction is adopted.

71. RULE.—*Conceive the sign of the subtrahend to be changed from + to − or from − to +, and then proceed as in addition.*

NOTE.—Students should test their work in subtraction by determining whether the sum of the remainder and subtrahend is equal to the minuend.

72. To subtract one term from another, the two terms being partly similar.

RULE.—*Indicate the subtraction of the dissimilar part in the subtrahend from the dissimilar part in the minuend, and write the result in a parenthesis as a compound coefficient of the similar part.*

EXAMPLES.

$$\begin{array}{r} ax \\ bx \\ \hline (a-b)x \end{array} \qquad \begin{array}{r} by \\ -cy \\ \hline (b+c)y \end{array} \qquad \begin{array}{r} ax^2 \\ x^2 \\ \hline (a-1)x^2 \end{array} \qquad \begin{array}{r} -y^2 \\ -by^2 \\ \hline (b-1)y^2 \end{array}$$

73. If a polynomial is inclosed by a parenthesis or other sign of grouping, and there is a minus sign before it, the parenthesis or other symbol may be removed by changing the signs of the terms in it.

If a polynomial is placed within a sign of grouping and preceded by a minus sign, the signs of the terms placed within the sign of grouping must be changed.

MULTIPLICATION.

74. Multiplication is the process of taking one quantity as many times as there are units in another.

75. The Multiplicand is the quantity to be taken.

76. The Multiplier is the quantity that shows how many times the multiplicand is to be taken.

77. The Product is the quantity obtained by multiplication.

78. The multiplicand and multiplier are factors of the product.

79. To perform multiplication.

b	$-b$	b	$-b$
$\underline{a}$	$\underline{a}$	$\underline{-a}$	$\underline{-a}$

The above examples represent all possible cases in multiplication with reference to the signs of the multiplier and multiplicand.

Multiplication is the process of taking one quantity as many times as there are units in another, or it may be defined as the process of uniting two or more equal quantities. The multiplicand represents one of the equal quantities, and the multiplier indicates how many of the equal quantities are to be united.

Uniting two b's, the result is $2b$; uniting three b's, the result is $3b$; uniting four b's, the result is $4b$; uniting five b's, the result is $5b$; uniting any number of b's, a representing any number, the result is ab. Hence the correct product of b and a is ab.

Uniting two $-b$'s, the result is $-2b$; uniting three $-b$'s, the result is $-3b$; uniting four $-b$'s, the result is $-4b$; uniting five $-b$'s, the result is $-5b$; uniting any number of $-b$'s, a representing any number, the result is $-ab$. Hence the correct product of $-b$ and a is $-ab$.

As has been proved in the first example, the product of b and a is ab. Since the multiplier in the third example is negative, the product must be of the opposite quality from what it would be if the multiplier were positive. Therefore the correct product of b and $-a$ is $-ab$.

As has been proved in the second example, the product of $-b$ and a is $-ab$. Since the multiplier in the fourth example is negative, the product must be of the opposite quality from what it would be if the multiplier were positive. Therefore the correct product of $-b$ and $-a$ is ab.

It appears from the above that when the multiplier and multiplicand have like signs, the product is positive; and when they have unlike signs, the product is negative.

By the application of this law to more than two quantities, it follows that:

The product of any number of positive quantities is positive.
The product of an even number of negative quantities is positive.
The product of an odd number of negative quantities is negative.

Since the product must contain all the factors of the multiplier and multiplicand, the numerical factor in the multiplicand must be multiplied by the numerical factor in the multiplier; and if the same letter is used in the two factors, it must appear as a factor as many times in the product as it is used in multiplier and multiplicand, and this will be indicated by the sum of its exponents.

$$5a \times 3a^2 = 5 \cdot 3 \cdot aaa = 15a^3.$$

$$7a^3x^2 \times 2a^2x = 7 \cdot 2 \cdot aaaxxaax = 14a^5x^3.$$

80. PRINCIPLES.—1. *The product of two quantities is positive when they have like signs.*

2. *The product of two quantities is negative when they have unlike signs.*

3. *The product of any number of positive quantities is positive.*

4. *The product of an even number of negative quantities is positive.*

5. *The product of an odd number of negative quantities is negative.*

81. To find the product of two or more monomials.

RULE. — *To the product of the numerical coefficients annex the literal quantities, giving each letter an exponent equal to the sum of the exponents of that letter in the several monomials. Give the result the proper sign.*

EXAMPLES.

$$\begin{array}{ccccc}
4a & -5b^2 & -4x^2y & 3ab^2 & -3a^2b^nc \\
3a & -2b & x^3y^2 & -2cd^3 & -2a^2cx^m \\
\hline
12a^2 & 10b^3 & -4x^5y^3 & -6ab^2cd^3 & 6a^4b^nc^2x^m
\end{array}$$

TO THE TEACHER. — After students have written the products in multiplication of monomials, require them to give the same rapidly at sight.

82. To multiply a polynomial by a monomial.

RULE. — *Multiply each term of the multiplicand by the multiplier according to the rule for multiplication of monomials.*

EXAMPLE.

$$\begin{array}{l}
5a^3b^2c - 4a^2b^3c + 3a^2b - 2b^2c \\
2a^2b^3c \\
\hline
10a^5b^5c^2 - 8a^4b^6c^2 + 6a^4b^4c - 4a^2b^5c^2
\end{array}$$

83. To multiply a polynomial by a polynomial.

RULE. — *Arrange the multiplicand and multiplier with reference to the same letter. Multiply each term of the multiplicand by each term of the multiplier, and add the partial products.*

EXAMPLES.

1.
$$\begin{array}{llllll}
4a^3b^2 & -\ 3a^2b^3 & +\ 2ab^4 & +\ 2b^5 & & \\
2a^2b & -\ 2ab^2 & +\ b^3 & & & \\
\hline
8a^5b^3 & -\ 6a^4b^4 & +\ 4a^3b^5 & +\ 4a^2b^6 & & \\
 & -\ 8a^4b^4 & +\ 6a^3b^5 & -\ 4a^2b^6 & -\ 4ab^7 & \\
 & & 4a^3b^5 & -\ 3a^2b^6 & +\ 2ab^7 & +\ 2b^8 \\
\hline
8a^5b^3 & -\ 14a^4b^4 & +\ 14a^3b^5 & -\ 3a^2b^6 & -\ 2ab^7 & +\ 2b^8
\end{array}$$

2.
$$\begin{array}{l} a - b + 2 \\ c + x - 3 \\ \hline ac - bc + 2c + ax - bx + 2x - 3a + 3b - 6 \end{array}$$

FORMULAS.

84. A Formula is an algebraic expression of a general rule or principle.

To the Teacher. — In teaching these formulas, it will be necessary to teach what is meant by the square of a quantity and how it is found.

85. First Formula. — $(a + b)^2 = a^2 + 2ab + b^2$.

Since a and b represent any two quantities, $(a + b)^2$ represents the square of the sum of any two quantities. The square of $a + b$ is found by multiplication to be $a^2 + 2ab + b^2$. This square is the square of the first quantity, plus twice the product of the two, plus the square of the second. Hence the following principle:

86. Principle. — *The square of the sum of two quantities is equal to the square of the first, plus twice the product of the two, plus the square of the second.*

87. Second Formula. — $(a - b)^2 = a^2 - 2ab + b^2$.

Since a represents any quantity and b any less quantity, $(a - b)^2$ represents the square of the difference of any two quantities. The square of $a - b$ is found by multiplication to be $a^2 - 2ab + b^2$. This square is the square of the first quantity, minus twice the product of the two, plus the square of the second. Hence the following principle:

88. Principle. — *The square of the difference of two quantities is equal to the square of the first, minus twice the product of the two, plus the square of the second.*

89. Third Formula. — $(a + b)(a - b) = a^2 - b^2$.

Since a represents any quantity and b any less quantity, $a + b$ represents the sum and $a - b$ the difference of any two quantities,

and $(a+b)(a-b)$ represents the product of the sum and difference of any two quantities. The product of $a+b$ and $a-b$ is found by multiplication to be a^2-b^2. This product is the square of the first quantity minus the square of the second. Hence the following principle:

90. PRINCIPLE. — *The product of the sum and difference of two quantities is equal to the square of the first minus the square of the second.*

PRODUCT OF TWO BINOMIALS WITH A COMMON TERM.

91. These products can be verified by multiplication:

1. $(x+4)(x+3)=x^2+7x+12.$
2. $(x-5)(x-3)=x^2-8x+15.$
3. $(x+7)(x-3)=x^2+4x-21.$
4. $(x+2)(x-9)=x^2-7x-18.$

The above examples are types of all those examples in multiplication in which it is required to find the product of two binomial factors having only one common term. They represent all possible conditions with reference to the signs and relative values of the second terms.

By carefully observing the number of terms in each product and how each term arises, the following laws are discovered, which enable the student to write the product of two such binomials by inspection:

Each product consists of three terms.

The first term of each product is the square of the common term.

The second term of each product is the product of the common term and the algebraic sum of the second terms.

The third term of each product is the product of the second terms.

When the signs of the second terms are alike, the second term of the product has the same sign; and the third term is positive.

When the signs of the second terms are unlike, the second

term of the product takes the sign of the second term which is numerically greater; and the third term is negative.

Hence the following principle:

92. PRINCIPLE. — *The product of any two binomials having a common term is composed of the square of the common term, the product of the common term and the algebraic sum of the second terms, and the product of the second terms.*

TO THE TEACHER. — After students have written the products under these formulas, require them to give the same rapidly at sight.

DIVISION.

93. Division is the process of finding how many times one quantity is contained in another. Or,

Division is the process of finding one of two factors when their product and the other factor are given.

Division is therefore the converse of multiplication, the dividend corresponding to the product, the divisor to the multiplicand, and the quotient to the multiplier.

94. The Dividend is the quantity to be divided.

95. The Divisor is the quantity by which another quantity is to be divided.

96. The Quotient is the quantity obtained by division.

97. To perform division.

Since division is the converse of multiplication, the dividend representing the product, the divisor one factor, and the quotient the other, the rule for division is easily deduced from the process of multiplication. By the definition of division, the product of the divisor and quotient must be equal to the dividend. This equality is the test for the correctness of any quotient.

In the derivation of the rule, three things must be determined: the sign of the quotient, the exponent of each letter in the quotient, and the coefficient of the quotient.

Signs.

$$+b\times+a=+ab,\quad\therefore\ +ab\div+a=+b.$$
$$+b\times-a=-ab,\quad\therefore\ -ab\div-a=+b.$$
$$-b\times-a=+ab,\quad\therefore\ +ab\div-a=-b.$$
$$-b\times+a=-ab,\quad\therefore\ -ab\div+a=-b.$$

The four divisions in the above work represent all possible cases in division with reference to the signs. The quotient in each case is correct, because, from the multiplication, the quotient and divisor are known to be the factors of the dividend. It appears that when dividend and divisor have like signs, the quotient is positive; and when they have unlike signs, the quotient is negative. Hence the law:

Like signs give plus, and unlike signs give minus.

Exponents.

$$a^3c^2x\times ac^3x^2=a^4c^5x^3,\quad\therefore\ a^4c^5x^3\div ac^3x^2=a^3c^2x.$$

In the preceding division, the quotient is correct, because, from the multiplication, the quotient and divisor are known to be the factors of the dividend.

Since the exponent of each letter in the dividend is the sum of the exponents of that letter in both factors, it is evident that the exponent of each letter in the quotient is found by subtracting the exponent of that letter in the divisor from the exponent of that letter in the dividend.

Coefficients.

$$5a^3\times3a^2=15a^5,\quad\therefore\ 15a^5\div3a^2=5a^3.$$

The coefficient of the quotient is found by dividing the coefficient of the dividend by the coefficient of the divisor.

98. Principles. — 1. *The quotient is positive when the dividend and divisor have like signs.*

2. *The quotient is negative when the dividend and divisor have unlike signs.*

3. *The exponent of each letter in the quotient is equal to*

the exponent of that letter in the dividend diminished by the exponent of that letter in the divisor.

4. *If any letter has the same exponent in dividend and divisor, the exponent of that letter in the quotient is zero, and the letter is not written in the quotient.*

5. *If a letter is found in the dividend and not in the divisor, it is brought down in the quotient with the same exponent.*

6. *If a letter is found in the divisor and not in the dividend, it is brought down in the quotient with the sign of its exponent changed.*

99. To divide one monomial by another.

RULE. — *Divide the numerical coefficient of the dividend by the numerical coefficient of the divisor, and to the quotient annex the literal quantities, giving each letter an exponent equal to the exponent of that letter in the dividend, diminished by the exponent of that letter in the divisor. Give the result the proper sign.*

EXAMPLES.

$$2a\overline{)8a^3} \qquad -3a^2b\overline{)12a^3b^2c} \qquad -2a^2bc\overline{)-8a^2b^4} \qquad 3a^nb\overline{)-9a^mb^n}$$
$$4a^2 \qquad\qquad -4abc \qquad\qquad 4b^3c^{-1} \qquad\qquad -3a^{m-n}b^{n-1}$$

100. To divide a polynomial by a monomial.

RULE. — *Divide each term of the dividend by the divisor according to the rule for division of monomials.*

EXAMPLE.

$$4ab^2c\overline{)16a^2b^4c - 12a^3b^2c^2 + 20ab^2c^4}$$
$$4ab^2 \quad - \quad 3a^2c \quad + \quad 5c^3$$

101. To divide a polynomial by a polynomial.

RULE. — *Arrange the dividend and divisor with reference to the same letter, writing the divisor at the right of the dividend.*

Divide the first term of the dividend by the first term of the divisor, and write the quotient under the divisor for the first term of the quotient.

Multiply the divisor by the first term of the quotient, and subtract the product from the dividend. To the remainder annex one or more terms of the dividend for a new dividend.

Divide the new dividend as before, and continue the process until there is no remainder or until the first term of the remainder is not divisible by the first term of the divisor.

If there is a remainder after the last division, write it over the divisor in the form of a fraction, and annex it with the proper sign to the part of the quotient already obtained.

EXAMPLES.

1.

$$
\begin{array}{l|l}
x^3 - 3x^2y + 3xy^2 - y^3 & x - y \\
\underline{x^3 - x^2y} & \overline{x^2 - 2xy + y^2} \\
\quad -2x^2y + 3xy^2 & \\
\quad \underline{-2x^2y + 2xy^2} & \\
\qquad\qquad xy^2 - y^3 & \\
\qquad\qquad \underline{xy^2 - y^3} &
\end{array}
$$

2.

$$
\begin{array}{l|l}
a^4 - a^2b^2 + 2ab^3 - b^4 & a^2 - ab + b^2 \\
\underline{a^4 - a^3b + a^2b^2} & \overline{a^2 + ab - b^2} \\
\qquad a^3b - 2a^2b^2 + 2ab^3 & \\
\qquad \underline{a^3b - a^2b^2 + ab^3} & \\
\qquad\qquad - a^2b^2 + ab^3 - b^4 & \\
\qquad\qquad \underline{- a^2b^2 + ab^3 - b^4} &
\end{array}
$$

3.

$$
\begin{array}{l|l}
x^5 + y^5 & x + y \\
\underline{x^5 + x^4y} & \overline{x^4 - x^3y + x^2y^2 - xy^3 + y^4} \\
\quad -x^4y + y^5 & \\
\quad \underline{-x^4y - x^3y^2} & \\
\qquad\qquad x^3y^2 + y^5 & \\
\qquad\qquad \underline{x^3y^2 + x^2y^3} & \\
\qquad\qquad\quad -x^2y^3 + y^5 & \\
\qquad\qquad\quad \underline{-x^2y^3 - xy^4} & \\
\qquad\qquad\qquad\quad xy^4 + y^5 & \\
\qquad\qquad\qquad\quad \underline{xy^4 + y^5} &
\end{array}
$$

4.
$$\begin{array}{l|l} x^5 - 1 & x - 1 \\ \cline{2-2} \underline{x^5 - x^4} & x^4 + x^3 + x^2 + x + 1 \\ \quad x^4 - 1 & \\ \quad \underline{x^4 - x^3} & \\ \qquad x^3 - 1 & \\ \qquad \underline{x^3 - x^2} & \\ \qquad\quad x^2 - 1 & \\ \qquad\quad \underline{x^2 - x} & \\ \qquad\qquad x - 1 & \\ \qquad\qquad \underline{x - 1} & \end{array}$$

5.
$$\begin{array}{l|l} x^4 + 4y^4 & x^2 - 2xy + 2y^2 \\ \cline{2-2} \underline{x^4 - 2x^3y + 2x^2y^2} & x^2 + 2xy + 2y^2 \\ \quad 2x^3y - 2x^2y^2 + 4y^4 & \\ \quad \underline{2x^3y - 4x^2y^2 + 4xy^3} & \\ \qquad 2x^2y^2 - 4xy^3 + 4y^4 & \\ \qquad \underline{2x^2y^2 - 4xy^3 + 4y^4} & \end{array}$$

ZERO EXPONENTS.

102. By the law of exponents in division,

$$x^3 \div x = x^2$$
$$x^5 \div x^2 = x^3$$
$$a^m \div a^n = a^{m-n}$$
$$a^n \div a^n = a^{n-n} = a^0$$
$$\text{But } a^n \div a^n = 1$$

Since a^0 and 1 are both equal to the same quantity, they are equal to each other, and $a^0 = 1$.

Since a represents any quantity, it follows that *any quantity with a zero exponent is equal to* 1.

Why is $a^0b^2c^3$ equal to b^2c^3?

Why is $4ax^0y^4$ equal to $4ay^4$?

SPECIAL CASES IN DIVISION.

103. ***The difference of the same powers of two quantities is exactly divisible by the difference of the quantities themselves.***

General Demonstration.

Let x represent any quantity, and y any smaller quantity, and n any positive integer. Then $x^n - y^n$ will represent the difference of the same powers of two quantities, and $x - y$ the difference of the quantities themselves.

$$
\begin{array}{l|l}
x^n - y^n & x - y \\
\underline{x^n - x^{n-1}y} & x^{n-1} + x^{n-2}y + x^{n-3}y^2 + x^{n-4}y^3 + x^{n-5}y^4 \\
\quad x^{n-1}y - y^n & \\
\quad \underline{x^{n-1}y - x^{n-2}y^2} & \\
\qquad x^{n-2}y^2 - y^n & \\
\qquad \underline{x^{n-2}y^2 - x^{n-3}y^3} & \\
\qquad\quad x^{n-3}y^3 - y^n & \\
\qquad\quad \underline{x^{n-3}y^3 - x^{n-4}y^4} & \\
\qquad\qquad x^{n-4}y^4 - y^n & \\
\qquad\qquad \underline{x^{n-4}y^4 - x^{n-5}y^5} & \\
\qquad\qquad\quad x^{n-5}y^5 - y^n &
\end{array}
$$

In the preceding division, each remainder consists of two terms. The second term of each remainder is $-y^n$. If the division is exact, there will finally be no remainder; and since $-y^n$ is constantly brought down after each division, in order that the remainder may finally be 0, it must take the form of $y^n - y^n$. The first term of the first remainder is $x^{n-1}y$; the first term of the second remainder is $x^{n-2}y^2$; the first term of the third remainder is $x^{n-3}y^3$; the first term of the fourth remainder is $x^{n-4}y^4$; the first term of the fifth remainder is $x^{n-5}y^5$; the first term of the nth remainder, n representing *any* positive integer, is $x^{n-n}y^n$, or x^0y^n, or $1 \times y^n$, or y^n.

The second term of each remainder being $-y^n$, the entire nth remainder is $y^n - y^n$, or 0, and the division is exact.

104. *The difference of the same even powers of two quantities is exactly divisible by both the difference and the sum of the quantities themselves.*

General Demonstration.

It has already been proved that the difference of the same powers of two quantities is exactly divisible by the difference of the quantities themselves, hence the difference of the same *even* powers is exactly divisible by the difference of the quantities.

Let x represent any quantity, and y any smaller quantity, and n any positive even integer. Then $x^n - y^n$ will represent the difference of the same even powers of two quantities, and $x + y$ the sum of the quantities themselves.

$$
\begin{array}{l|l}
x^n - y^n & x + y \\
\underline{x^n + x^{n-1}y} & \overline{x^{n-1} - x^{n-2}y + x^{n-3}y^2 - x^{n-4}y^3 + x^{n-5}y^4} \\
\quad - x^{n-1}y - y^n & \\
\quad \underline{- x^{n-1}y - x^{n-2}y^2} & \\
\qquad\qquad x^{n-2}y^2 - y^n & \\
\qquad\qquad \underline{x^{n-2}y^2 + x^{n-3}y^3} & \\
\qquad\qquad\qquad - x^{n-3}y^3 - y^n & \\
\qquad\qquad\qquad \underline{- x^{n-3}y^3 - x^{n-4}y^4} & \\
\qquad\qquad\qquad\qquad x^{n-4}y^4 - y^n & \\
\qquad\qquad\qquad\qquad \underline{x^{n-4}y^4 - x^{n-5}y^5} & \\
\qquad\qquad\qquad\qquad\qquad x^{n-5}y^5 - y^n &
\end{array}
$$

In the preceding division, each remainder consists of two terms. The second term of each remainder is $-y^n$. If the division is exact, there will finally be no remainder; and since $-y^n$ is constantly brought down after each division, in order that the remainder may finally be 0, it must take the form of $y^n - y^n$. The first term of the first remainder is $-x^{n-1}y$; the first term of the second remainder is $+x^{n-2}y^2$; the first term of the third remainder is $-x^{n-3}y^3$; the first term of the fourth remainder is $+x^{n-4}y^4$; the first term of the fifth remainder is

$-x^{n-5}y^5$. The first term of the odd remainders is negative, and the first term of the even remainders is positive. Therefore the first term of the nth remainder, n representing *any* positive even integer, is $x^{n-n}y^n$, or x^0y^n, or $1 \times y^n$, or y^n.

The second term of each remainder being $-y^n$, the entire nth remainder is $y^n - y^n$, or 0, and the division is exact.

105. ***The difference of the same odd powers of two quantities is not exactly divisible by the sum of the quantities themselves.***

As has been shown in the preceding demonstration, the first term of the odd remainders in this division is negative. Hence the first term of the nth remainder in this division, n representing *any* positive odd integer, is $-x^{n-n}y^n$, or $-x^0y^n$, or $-1 \times y^n$, or $-y^n$.

The second term of each remainder being $-y^n$, the entire nth remainder is $-y^n - y^n$, or $-2y^n$. This monomial remainder is not divisible by the binomial divisor $x - y$, therefore the division is not exact.

106. ***The sum of the same odd powers of two quantities is exactly divisible by the sum of the quantities themselves.***

General Demonstration.

Let x represent any quantity, and y any other quantity, and n any positive odd integer. Then $x^n + y^n$ will represent the sum of the same odd powers of two quantities, and $x + y$ the sum of the quantities themselves.

$$
\begin{array}{l|l}
x^n + y^n & x + y \\
\hline
\underline{x^n + x^{n-1}y} & x^{n-1} - x^{n-2}y + x^{n-3}y^2 - x^{n-4}y^3 + x^{n-5}y^4 \\
\quad - x^{n-1}y + y^n & \\
\quad \underline{- x^{n-1}y - x^{n-2}y^2} & \\
\qquad\qquad x^{n-2}y^2 + y^n & \\
\qquad\qquad \underline{x^{n-2}y^2 + x^{n-3}y^3} & \\
\qquad\qquad\qquad - x^{n-3}y^3 + y^n & \\
\qquad\qquad\qquad \underline{- x^{n-3}y^3 - x^{n-4}y^4} & \\
\qquad\qquad\qquad\qquad x^{n-4}y^4 + y^n & \\
\qquad\qquad\qquad\qquad \underline{x^{n-4}y^4 + x^{n-5}y^5} & \\
\qquad\qquad\qquad\qquad\qquad - x^{n-5}y^5 - y^n &
\end{array}
$$

In the preceding division, each remainder consists of two terms. The second term of each remainder is $+ y^n$. If the division is exact, there will finally be no remainder; and since $+ y^n$ is constantly brought down after each division, in order that the remainder may finally be 0, it must take the form of $- y^n + y^n$. The first term of the first remainder is $- x^{n-1}y$; the first term of the second remainder is $+ x^{n-2}y^2$; the first term of the third remainder is $- x^{n-3}y^3$; the first term of the fourth remainder is $+ x^{n-4}y^4$; the first term of the fifth remainder is $- x^{n-5}y^5$. The first term of the odd remainders is negative, and the first term of the even remainders is positive. Therefore the first term of the nth remainder, n representing *any* positive odd integer, is $- x^{n-n}y^n$, or $- x^0y^n$, or $- 1 \times y^n$, or $- y^n$.

The second term of each remainder being $+ y^n$, the entire nth remainder is $- y^n + y^n$, or 0, and the division is exact.

107. *The sum of the same even powers of two quantities is not exactly divisible by the sum of the quantities themselves.*

As has been shown in the preceding demonstration, the first term of the even remainders in this division is positive. Hence the first term of the nth remainder in this division, n representing *any* positive even integer, is $x^{n-n}y^n$, or x^0y^n, or $1 \times y^n$, or y^n.

The second term of each remainder being $+ y^n$, the entire nth remainder is $y^n + y^n$, or $2 y^n$. This monomial remainder is not divisible by the binomial divisor $x + y$, therefore the division is not exact.

108. *The sum of the same powers of two quantities is not exactly divisible by the difference of the quantities themselves.*

By performing the division, as in the preceding cases, it can be shown that the first term of each remainder is positive.

The second term of each remainder being $+ y^n$, the entire nth remainder is $y^n + y^n$, or $2 y^n$. This monomial remainder is not divisible by the binomial divisor $x - y$, therefore the division is not exact.

The Quotients.

109. The number of terms in the quotient is n; that is, it is equal to the exponent of each letter in the dividend divided by the exponent of that letter in the divisor.

x is found in every term of the quotient except the last, and y is found in every term except the first.

The exponent of x in the first term of the quotient is 1 less than the exponent of x in the dividend, and diminishes by 1 in each succeeding term.

The exponent of y in the second term of the quotient is 1, and increases by 1 in each succeeding term.

When the divisor is the difference of two quantities, all the terms of the quotient are positive.

When the divisor is the sum of two quantities, the odd terms of the quotient are positive, and the even terms are negative.

NOTE.—The exponent of x in the quotient diminishes by 1 and the exponent of y increases by 1 only when the exponents of x and y in the divisor are 1.

It is important to remember that the exponent of x in the quotient diminishes by the exponent of x in the divisor, and the exponent of y increases by the exponent of y in the divisor, *whatever these exponents may be.*

EXAMPLES.

$$(x^3+y^3)\div(x+y)=x^2-xy+y^2.$$
$$(a^3-x^3)\div(a-x)=a^2+ax+x^2.$$
$$(x^3-y^6)\div(x-y^2)=x^2+xy^2+y^4.$$
$$(a^9+x^3)\div(a^3+x)=a^6-a^3x+x^2.$$
$$(x^6+y^9)\div(x^2+y^3)=x^4-x^2y^3+y^6.$$

Special attention should be given to binomials one term of which is a number.

$x^3+1=x^3+1^3.$ $\qquad$ $8-x^3=2^3-x^3.$

$27+x^3=3^3+x^3.$ $\qquad$ $x^3-64=x^3-4^3.$

$$(x^3+1)\div(x+1)=x^2-x+1.$$
$$(8-x^3)\div(2-x)=4+2x+x^2.$$
$$(27+x^3)\div(3+x)=9-3x+x^2.$$
$$(x^3-64)\div(x-4)=x^2+4x+16.$$

FACTORING.

110. The **Factors** of a quantity are the quantities which multiplied together will give the quantity.

111. **Factoring** is the process of separating a quantity into its factors.

112. Case I. — **To resolve a polynomial into two factors, one of which is a monomial and the other a polynomial.**

113. Any polynomial having a common factor in every term can be factored by this case. The monomial factor is determined by inspection.

114. Rule. — *Divide the polynomial by the monomial factor. The divisor and quotient will be the factors.*

EXAMPLES.

1. $9a^3b^2c - 12a^2b^3c^2 + 15a^4b^4c^3 = 3a^2b^2c(3a - 4bc + 5a^2b^2c^2)$.

2. $8x^2y^3 - 12x^3y^2 + 16x^2y^2 - 4xy = 4xy(2xy^2 - 3x^2y + 4xy - 1)$.

Note. — Whenever the monomial factor is the same as one of the terms of the polynomial, as in the second example given above, one of the terms of the polynomial factor will be 1.

115. The terms of a polynomial can sometimes be grouped so as to show a common compound factor. Such polynomials usually contain four or six terms. The terms may be grouped in any manner, provided each group shows the same compound factor. The first term is generally taken with the second, and the third with the fourth; but the first may be taken with the third, and the second with the fourth.

$$ax + ay + bx + by$$

The first and second terms of this polynomial contain the common factor a, and the third and fourth terms contain

the common factor b. Grouping the terms of the polynomial in this manner, and factoring each group, the polynomial assumes the following form:

$$a(x+y)+b(x+y)$$

By the use of the sign of grouping, the polynomial has been reduced to two terms. These terms are similar with reference to the compound factor. Combining these terms according to the rule for addition of terms partly similar, the result is $(a+b)(x+y)$. Hence,

1. $ax+ay+bx+by=a(x+y)+b(x+y)=(a+b)(x+y)$

In like manner,

2. $ac-ad+bc-bd=a(c-d)+b(c-d)=(a+b)(c-d)$

In the two preceding examples, a positive monomial factor is taken out of each group. It must be remembered that the polynomial can not be factored in this manner, unless the compound factor is the same in each group. Notice the following examples:

3. $ax+ay-bx-by=a(x+y)-b(x+y)=(a-b)(x+y)$
4. $ax-ay-bx+by=a(x-y)-b(x-y)=(a-b)(x-y)$

It is plain that to make the compound factor in the third and fourth terms the same as the compound factor in the first and second terms, the negative factor $-b$ must be taken out of the second group in each example.

Sometimes the compound factor in one group is like the two remaining terms, or like those terms with their signs changed. When this is the case, the monomial factor taken out of one group is $+1$ or -1. The following examples should be carefully studied:

5. $ax-ay+x-y=a(x-y)+1(x-y)=(a+1)(x-y)$
6. $ax-ay-x+y=a(x-y)-1(x-y)=(a-1)(x-y)$
7. $1-b-a+ab=1(1-b)-a(1-b)=(1-a)(1-b)$

In this work in factoring, the signs of the terms should be carefully noticed.

If the sign of the fourth term in any one of these examples were different, the polynomial could not be factored.

NOTE. — Beginning on page 61, prime quantities are occasionally given in the exercises in factoring. This is done that students may find it necessary to examine the quantities to *know* whether they are prime or composite.

TO THE TEACHER. — As roots of monomials have to be found in the following cases in factoring, develop here the idea of root, square root, cube root, etc. Teach also how to find any root of a monomial.

116. CASE II. — **To factor a trinomial square.**

By the first and second formulas in multiplication,

$$(x+1)(x+1)=x^2+2x+1$$
$$(x-1)(x-1)=x^2-2x+1$$
$$(x+y)(x+y)=x^2+2xy+y^2$$
$$(x-y)(x-y)=x^2-2xy+y^2$$

The trinomials obtained by these formulas are called *trinomial squares*, and the work of factoring them consists in determining of what two binomials they are the product.

117. **A Trinomial Square** is a trinomial one of whose terms is twice the product of the square roots of the other two.

$2ab+a^2+b^2 \qquad 4x^2+y^2-4xy \qquad x^4+1+2x^2$

118. **An Arranged Trinomial Square** is a trinomial square in which the second term is twice the product of the square roots of the other two.

$a^2+2ab+b^2 \qquad 4x^2-4xy+y^2 \qquad x^4+2x^2+1$

119. To determine whether a given trinomial is a trinomial square, look first for two squares, which are always positive. If two of the terms are squares, find the square root of each, multiply one root by the other, and that product

by two. If the product is the remaining term of the trinomial, the trinomial is a trinomial square.

NOTE.—A trinomial square can have only one negative term in it, and that term is the one which is twice the product of the square roots of the other two.

120. RULE.—*Consider the trinomial an arranged trinomial square. Find the square roots of the first and last terms, connect these square roots with the sign of the second term, and write the binomial twice as a factor.*

EXAMPLES.

1. $x^2 + 2xy + y^2 = (x+y)(x+y)$
2. $a^2 - 8a + 16 = (a-4)(a-4)$
3. $x^4 + 9 + 6x^2 = (x^2+3)(x^2+3)$

TO THE TEACHER.—Show that the factors of $a^2 - 8a + 16$ are $a - 4$ and $a - 4$, or $4 - a$ and $4 - a$. If the trinomial be arranged as above, the factors will be $a - 4$ and $a - 4$. If the first and last terms be interchanged, the factors will be $4 - a$ and $4 - a$.

Give much oral drill in factoring, and require students to give the factors rapidly at sight.

121. CASE III.—**To factor the difference of two squares.**

By the third formula in multiplication,

$$(x+1)(x-1) = x^2 - 1$$
$$(x+2)(x-2) = x^2 - 4$$
$$(x+y)(x-y) = x^2 - y^2$$

The work of factoring such binomials is evidently the reverse of the multiplication, and the factors are the sum and difference of the square roots of the two terms of the binomial.

122. RULE.—*Find the square roots of the two terms of the binomial, and write for the factors the sum and difference of the square roots.*

EXAMPLES.

1. $x^2 - y^2 = (x+y)(x-y)$
2. $a^2 - 4 = (a+2)(a-2)$

3. $x^2 - 1 = (x+1)(x-1)$
4. $1 - x^2 = (1+x)(1-x)$
5. $a^4 - b^2 = (a^2+b)(a^2-b)$

123. By factoring the difference of the square roots each time, the work can be continued indefinitely. It is generally continued as far as it can be without getting fractional exponents.

EXAMPLES.

1. $x^8 - 1 = (x^4+1)(x^2+1)(x+1)(x-1)$
2. $a^8 - b^8 = (a^4+b^4)(a^2+b^2)(a+b)(a-b)$

124. In the examples thus far given in this case, both of the squares have been monomials. One of the squares may be a trinomial and the other a monomial, or both of the squares may be trinomials.

The trinomial squares may be expressed with or without a sign of grouping, as follows:

$$(a-b)^2 - c^2 = a^2 - 2ab + b^2 - c^2$$
$$a^2 - (b+c)^2 = a^2 - b^2 - 2bc - c^2$$
$$(a-c)^2 - (x-y)^2 = a^2 - 2ac + c^2 - x^2 + 2xy - y^2$$

The trinomial square in the second example, if left in the parenthesis, is $b^2 + 2bc + c^2$; but when the parenthesis is removed, all the signs are changed. The second trinomial square in the third example, if left in the parenthesis, is $x^2 - 2xy + y^2$; but when the parenthesis is removed, all the signs are changed. Before factoring such expressions, it is convenient to express each trinomial square as the square of a binomial. Great care must be exercised in doing this. When the first square is a trinomial, this change is made without any change in signs, for the parenthesis is not preceded by a minus sign; but when the second square is a trinomial, the sign of each term must be changed, for the parenthesis is preceded by the minus sign.

If a polynomial is the difference of two squares, it usually contains four or six terms; four, if one of the squares is a monomial and the other a trinomial; six, if both of the squares are trinomials. The term that contains two letters generally indicates what terms are taken with it to form a trinomial square. If three of the terms make a trinomial square as they are given in the example, this square must be placed first. If the remaining monomial square is negative, or if the remaining three terms, *with their signs changed*, make a trinomial square, the polynomial is the difference of two squares.

EXAMPLES.

1. $a^2+2ab+b^2-c^2=(a+b)^2-c^2=(a+b+c)(a+b-c)$
2. $a^2-b^2-2bc-c^2=a^2-(b+c)^2=(a+b+c)(a-b-c)$
3. $a^2+b^2-c^2-d^2-2ab+2cd=(a-b)^2-(c-d)^2$
$$=(a-b+c-d)(a-b-c+d)$$

125. CASE IV. — **To resolve the difference of the same odd powers of two quantities into two factors, one of which is a binomial and the other a polynomial.**

This case is an inference from one of the special cases in division. It has been demonstrated that the difference of the same odd powers of two quantities is exactly divisible by the difference of the quantities themselves.

Since the divisor and quotient are factors of the dividend, such binomials are resolvable into two factors, one of which is the divisor and the other the quotient.

EXAMPLES.

1. $x^3-y^3=(x-y)(x^2+xy+y^2)$
2. $x^6-y^3=(x^2)^3-y^3=(x^2-y)(x^4+x^2y+y^2)$
3. $x^3-y^6=x^3-(y^2)^3=(x-y^2)(x^2+xy^2+y^4)$
4. $8x^3-27=2^3x^3-3^3=(2x-3)(2^2x^2+2x3+3^2)$
$$=(2x-3)(4x^2+6x+9)$$

126. The binomial factor is the difference of the same odd roots of the two terms of the binomial.

All the coefficients in the polynomial factor are 1.

The exponent of the first term diminishes by the exponent of the first term of the binomial factor, and the exponent of the second term increases by the exponent of the second term of the binomial factor.

All the terms of the polynomial factor are positive.

127. From the above examples, it appears that the binomial factor is determined by the number or numbers that will divide the exponents in the given binomial.

If the exponents are divisible by 3, the binomial factor is the difference of the cube roots; if the exponents are divisible by 5, the binomial factor is the difference of the fifth roots; if the exponents are divisible both by 3 and 5, the binomial factor is the difference either of the cube roots or the fifth roots.

When the exponents are divisible by only one odd number, there will be only one set of factors; when the exponents are divisible by two odd numbers, there will be two sets of factors.

128. CASE V. — **To resolve the sum of the same odd powers of two quantities into two factors, one of which is a binomial and the other a polynomial.**

This case is also an inference from one of the special cases in division. It has been demonstrated that the sum of the same odd powers of two quantities is exactly divisible by the sum of the quantities themselves.

Since the divisor and quotient are factors of the dividend, such binomials are resolvable into two factors, one of which is the divisor and the other the quotient.

EXAMPLES.

1. $x^3 + 1 = (x + 1)(x^2 - x + 1)$
2. $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

3. $x^6 + y^6 = (x^2)^3 + y^3 = (x^2 + y)(x^4 - x^2y + y^2)$

4. $x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)$

5. $64x^3 + 125 = 4^3x^3 + 5^3 = (4x + 5)(4^2x^2 - 4x5 + 5^2)$
$$= (4x + 5)(16x^2 - 20x + 25)$$

129. The binomial factor is the sum of the same odd roots of the two terms of the binomial.

The coefficients and exponents in the polynomial factor are the same as in Case IV.

The odd terms of the polynomial factor are positive, and the even terms are negative.

130. When the exponents in the given binomial are divisible by only one odd number, there will be only one set of factors; when the exponents are divisible by two odd numbers, there will be two sets of factors.

131. CASE VI. — **To factor a quadratic trinomial.**

132. **A Quadratic Trinomial** is a trinomial one of whose terms is the product of the square root of one of the other terms and the sum of the factors of the remaining term.

$a^2 + 7a + 12$ $\qquad\qquad$ $a^2 - 6a + 8$

$x^2 + 2x - 15$ $\qquad\qquad$ $x^2 - 2x - 3$

133. RULE. — *Arrange the trinomial in the form of $x^2 + ax + b$. Find the square root of the first term for the first term of each binomial factor. Resolve the last term into two factors whose algebraic sum is a, and write these two factors, with their respective signs, for the second terms of the binomial factors.*

EXAMPLES.

1. $x^2 + 5x + 6 = (x + 3)(x + 2)$
2. $x^2 - 5x + 6 = (x - 3)(x - 2)$

3. $x^2 + 2x - 8 = (x+4)(x-2)$
4. $x^2 - 2x - 8 = (x+2)(x-4)$
5. $x^2 + 4x + 3 = (x+3)(x+1)$
6. $x^2 - 5x + 4 = (x-4)(x-1)$
7. $x^2 + 4x - 5 = (x+5)(x-1)$
8. $x^2 - 2x - 3 = (x-3)(x+1)$

134. Other trinomials can sometimes be resolved into two binomial factors. The factors are found by trial.

To the Teacher. — Have students multiply $2x+3$ by $3x+2$, $4x-3$ by $2x-4$, $3x+4$ by $2x-5$, $2x+3$ by $4x-3$, $6x+1$ by $x-4$, and $x+8$ by $4x-2$. Remind them that the multiplier and multiplicand are factors of the product. From an examination of the several products, lead them to see how the product is formed from the factors in each case. Now, suppose it is required to find the factors of $4x^2 + 22x - 12$. It is plain that $2x$ and $2x$ are the factors of $4x^2$, and that -4 and 3 are the factors of -12. It is then to be determined whether these factors will give the second term of the trinomial. The trial is made as follows:

$$\begin{array}{r} 2x-4 \\ 2x+3 \\ \hline -8x \\ 6x \\ \hline -2x \end{array}$$

It is necessary to multiply only for the second term. These factors give $-2x$ for the second term. Next try $2x-3$ and $2x+4$. These factors give $2x$ for the second term. By continued trial, it is found that the factors are $x+6$ and $4x-2$. Sufficient practice will give the power to determine very quickly the factors of such trinomials.

Note 1. — If the third term of the trinomial is positive, the second terms of the binomial factors have the same sign as the second term of the trinomial.

Note 2. — If the third term of the trinomial is negative, the second terms of the binomial factors have unlike signs.

135. Case VII. — **To resolve a trinomial into two trinomial factors.**

136. Trinomials to which some square must be added to make them trinomial squares can be factored by this case.

Q

137. Rule. — *Find what square must be added to the trinomial to make it a trinomial square. Add this square to the trinomial and also indicate the subtraction of it from the trinomial square thus formed. The resulting polynomial is the difference of two squares, and it can be factored by Case III.*

EXAMPLE.

$$\begin{array}{l} x^4 + 2x^2y^2 + 9y^4 \\ \qquad 4x^2y^2 \\ \hline x^4 + 6x^2y^2 + 9y^4 - 4x^2y^2 = (x^2 + 3y^2 + 2xy)(x^2 + 3y^2 - 2xy) \end{array}$$

138. When the second term of the trinomial is negative, two different squares can in most cases be added to the trinomial, thus making two different trincmial squares. One of the squares added is numerically less than the second term, and the other is numerically greater. When this can be done, there will be two sets of factors.

EXAMPLE.

$$\begin{array}{l} 4a^4 - 5a^2b^2 + b^4 \\ \qquad\quad a^2b^2 \\ \hline 4a^4 - 4a^2b^2 + b^4 - a^2b^2 = (2a^2 - b^2 + ab)(2a^2 - b^2 - ab) \end{array}$$

$$\begin{array}{l} 4a^4 - 5a^2b^2 + b^4 \\ \qquad 9a^2b^2 \\ \hline 4a^4 + 4a^2b^2 + b^4 - 9a^2b^2 = (2a^2 + b^2 + 3ab)(2a^2 + b^2 - 3ab) \end{array}$$

To the Teacher. — The very thorough review of factoring includes examples in all the different cases.

Students should be trained to approach this work systematically. Skill in factoring depends on the power to determine quickly how a quantity can be factored, if at all. The number of terms in the expression limits it to certain cases. If it is the sum of two quantities, it is limited to one case; if it is the difference of two quantities, it is limited to two cases; if it is a trinomial, it is limited to three cases.

When the quantity contains a monomial factor, that factor should

in every case be removed first. The remaining compound factor can often be resolved into two or more factors.

All binomials that are the difference of the same even powers of two quantities should first be resolved into the sum and difference of their square roots by Case III. It will then appear that one or both of these factors can be factored. The factors of $a^6 - x^6$ are $a^3 + x^3$ and $a^3 - x^3$. These can both be factored; the first by Case V, and the second by Case IV.

Students should be careful never to write a compound factor as one of the factors of the quantity, until they are sure it is prime. In the example above referred to, if $a^3 + x^3$ and $a^3 - x^3$ are first written and then the factors of each of these binomials written, it is plain that there would be two sets of factors, which would probably appear as follows:

$$(a^3 + x^3)(a^3 - x^3)(a + x)(a^2 - ax + x^2)(a - x)(a^2 + ax + x^2)$$

Students will avoid this common error by carefully observing the above direction.

The greatest possible skill in factoring is of the utmost importance, and it is recommended that during the whole course in algebra frequent exercises in factoring be given.

COMMON DIVISORS.

139. **A Common Divisor** of two or more quantities is a quantity that will divide each of them without a remainder.

140. Two quantities are prime to each other when they have no common factor except 1.

141. **The Highest Common Divisor** of two or more quantities is the quantity of the highest degree that will divide each of them without a remainder.

142. PRINCIPLES.—1. *Every factor of a quantity is a divisor of that quantity.*

2. *A common factor of two or more quantities is a common divisor of those quantities.*

3. *The highest common divisor of two or more quantities is the product of all their common factors.*

4. *A common divisor of two quantities is a divisor of their sum and also of their difference.*

5. *A divisor of a quantity is a divisor of any multiple of that quantity, and a common divisor of two quantities is a divisor of any multiple of either of them.*

6. *The highest common divisor of two quantities is not affected by multiplying or dividing either of them by a quantity that is not a factor of the other.*

Highest Common Divisor by Factoring.

143. To find the highest common divisor by factoring.

Rule. — *Resolve the quantities into their prime factors, and find the product of all the common factors.*

EXAMPLES.

1.
$$\begin{aligned} 18a^2b^5c &= 3 \times 3 \times 2 \times aa \times bbbbb \times c \\ 30a^3b^4c &= 3 \times 2 \times 5 \times aaa \times bbbb \times c \\ 42a^5b^2c &= 2 \times 3 \times 7 \times aaaaa \times bb \times c \\ 12a^4b^3c &= 2 \times 3 \times 2 \times aaaa \times bbb \times c \end{aligned}$$

$\therefore$ the H. C. D. is $3 \times 2 \times aa \times bb \times c$, or $6a^2b^2c$.

2.
$$\begin{aligned} 6acx - 3acy &= 3ac(2x - y) \\ 8cdx - 4cdy &= 4cd(2x - y) \\ 4bcx - 2bcy &= 2bc(2x - y) \end{aligned}$$

$\therefore$ the H. C. D. is $c(2x - y)$, or $2cx - cy$.

3.
$$\begin{aligned} x^2 - 8x + 16 &= (x - 4)(x - 4) \\ x^2 - 7x + 12 &= (x - 4)(x - 3) \\ x^2 + 5x - 36 &= (x - 4)(x + 9) \end{aligned}$$

$\therefore$ the H. C. D. is $x - 4$.

4.
$$\begin{aligned} 4x^2 - 24x + 36 &= 4(x - 3)(x - 3) \\ 6x^2 - 42x + 72 &= 6(x - 3)(x - 4) \\ 2x^2 + 12x - 54 &= 2(x - 3)(x + 9) \end{aligned}$$

$\therefore$ the H. C. D. is $2(x - 3)$, or $2x - 6$.

Highest Common Divisor by Continued Division.

144. When polynomials can not be factored readily by inspection, their highest common divisor may be found by the method of *continued division.*

This method, as well as the method by factoring, depends on the principles given in Art. 142, especially 4, 5, and 6, with which students should be very familiar.

145. The highest common divisor of numbers is generally found by continued division. This method will first be explained in the process of finding the highest common divisor of two numbers, and then it will be shown that a similar rule applies in finding the highest common divisor of *any* two numbers, or of any two algebraic quantities.

```
357)425(1
    357
    ---
     68)357(5
        340
        ---
         17)68(4
            68
            --
```

It is required to find the highest common divisor of 357 and 425.

The greater number is divided by the less; the first divisor is divided by the first remainder; the second divisor is divided by the second remainder; and this process is continued until there is no remainder. It will be shown that the last divisor is the H. C. D. sought.

Since every common divisor of two numbers is a divisor of their difference, a common divisor of 357 and 425 is a divisor of 68. A common divisor of 357 and 425 is therefore a common divisor of 68 and 357.

A divisor of 68 is a divisor of 5 times 68, or 340, and a common divisor of 68 and 357 is also a common divisor of 357 and 340. Since a common divisor of two numbers is a divisor of their difference, a common divisor of 357 and 340 is a divisor of 17. A common divisor of 68 and 357 is therefore a common divisor of 17 and 68. It has been shown that a common divisor of 357 and 425 is a common divisor of

68 and 357, and that a common divisor of 68 and 357 is a common divisor of 17 and 68. The H. C. D. sought can not therefore be greater than 17.

Since 17 is a factor of 68, it is a factor of 5 times 68, or 340.

Since 17 is a common factor of 17 and 340, it is a factor of their sum, or 357.

Since 17 is a common factor of 68 and 357, it is a factor of their sum, or 425.

Hence 17 is a common factor, and therefore a common divisor, of 357 and 425.

Since 17 is a common divisor of 357 and 425, and since it has been proved that the H. C. D. sought can not be greater than 17, 17 is the H. C. D. of 357 and 425.

146. Another illustration, in which the common factor appears throughout the division, may be helpful.

Let $11x$ and $14x$ represent any two numbers of which x is the only common factor.

$$\begin{array}{rllll}
11x) & 14x(1 & & & \\
 & \underline{11x} & & & \\
 & 3x) & 11x(3 & & \\
 & & \underline{9x} & & \\
 & & 2x) & 3x(1 & \\
 & & & \underline{2x} & \\
 & & & x) & 2x(2 \\
 & & & & \underline{2x}
\end{array}$$

Since x is the only common factor of these numbers, it is their H. C. D. The process shows how the factor *not common* gradually diminishes in the course of the division, and how the highest common divisor x appears in every divisor, dividend, and remainder, and finally appears alone as the last divisor.

General Demonstration.

147. It now remains to be proved that this method will give the highest common divisor of any two numbers, or of any two algebraic quantities.

Let A and B represent any two numbers of which B is the greater, or any two algebraic quantities of which B is the same degree as A, or higher degree than A.

B is divided by A, giving a quotient C and a remainder D. The first divisor is divided by the first remainder, giving a quotient E and a remainder F. The second divisor is divided by the second remainder, giving a quotient G and a remainder H. The third divisor is divided by the third remainder, giving a quotient I and no remainder. It will be shown that the last divisor is the H. C. D. sought.

$$\begin{array}{llllll}
A) & B\,(C & & & & \\
 & \underline{AC} & & & & \\
 & D) & A\,(E & & & \\
 & & \underline{DE} & & & \\
 & & F) & D\,(G & & \\
 & & & \underline{FG} & & \\
 & & & H) & F\,(I & \\
 & & & & \underline{HI} &
\end{array}$$

A factor of A is a factor of C times A, or AC, and a common factor of A and B is also a common factor of B and AC, and therefore a factor of D. A common factor of A and B is therefore a common factor of D and A.

A factor of D is a factor of E times D, or DE, and a common factor of D and A is also a common factor of A and DE, and therefore a factor of F. A common factor of D and A is therefore a common factor of F and D.

A factor of F is a factor of G times F, or FG, and a common factor of F and D is also a common factor of D and FG, and therefore a factor of H. A common factor of F and D is therefore a common factor of H and F.

It has been shown that a common factor of A and B is a common factor of D and A; that a common factor of D and A is a common factor of F and D; and that a common factor of F and D is a common factor of H and F. The H. C. D. sought can not therefore be greater, or of a higher degree, than H.

Since H is a factor of F, it is a factor of G times F, or FG.

Since H is a common factor of H and FG, it is a factor of their sum, or D.

Since H is a factor of D, it is a factor of E times D, or DE.

Since H is a common factor of F and DE, it is a factor of their sum, or A.

Since H is a factor of A, it is a factor of C times A, or AC.

Since H is a common factor of D and AC, it is a factor of their sum, or A.

Hence H is a common factor, and therefore a common divisor, of A and B.

Since H is a common divisor of A and B, and since it has been proved that the H. C. D. sought can not be greater than H, H is the H. C. D. of A and B.

EXAMPLES.

1.

$$
\begin{array}{r|l}
4x^3 - 4x^2 - 7x + 6 & 2x^2 - 5x + 3 \\
\cline{2-2}
4x^3 - 10x^2 + 6x & 2x + 3 \\
\cline{1-1}
6x^2 - 13x + 6 & \\
6x^2 - 15x + 9 & \\
\cline{1-1}
\end{array}
$$

$$
\begin{array}{r}
2x - 3)2x^2 - 5x + 3(x - 1 \\
2x^2 - 3x \\
\hline
-2x + 3 \\
-2x + 3 \\
\hline
\end{array}
$$

$\therefore$ the H. C. D. is $2x - 3$.

2.

$$
\begin{array}{r|l}
8x^3 - 8x^2 + 12x - 5 & 4x^2 + 4x - 3 \\
\cline{2-2}
8x^3 + 8x^2 - 6x & 2x - 4 \\
\cline{1-1}
-16x^2 + 18x - 5 & \\
-16x^2 - 16x + 12 & \\
\cline{1-1}
17)34x - 17 & \\
\end{array}
$$

$$
\begin{array}{r}
2x - 1)4x^2 + 4x - 3(2x - 3 \\
4x^2 - 2x \\
\hline
6x - 3 \\
6x - 3 \\
\hline
\end{array}
$$

$\therefore$ the H. C. D. is $2x - 1$.

Note 1. — It has appeared in the general demonstration that the highest common divisor sought is a common factor of each remainder, or divisor, and its corresponding dividend. Therefore, a factor of any remainder, which is not a factor of the dividend, may be removed; for, as it is not a *common* factor of the divisor and dividend, it can not be a factor of the highest common divisor. The remainder, $34x - 17$, in the second example, contains the factor 17, which is removed by division. There is often developed in the remainder by the multiplications and subtractions a factor which is not a factor of the dividend, and it is generally desirable to remove this factor.

3.
$$
\begin{array}{l}
6x^3 - 11x^2 - 2x + 8 \;\big|\; \underline{6x^2 - 11x + 4} \\
\underline{6x^3 - 11x^2 + 4x} \quad\;\; \big|\; x \\
\qquad -2\underline{)-6x + 8} \\
\qquad\qquad 3x - 4)6x^2 - 11x + 4(2x - 1 \\
\qquad\qquad\qquad\quad \underline{6x^2 - \;\; 8x} \\
\qquad\qquad\qquad\qquad\; -\;3x + 4 \\
\qquad\qquad\qquad\qquad\; \underline{-\;3x + 4}
\end{array}
$$

$\therefore$ the H. C. D. is $3x - 4$.

Note 2. — The first remainder in the third example contains the factor 2, which is not a factor of the dividend. If the factor 2 were removed, the divisor would be $-3x + 4$. This is exactly contained in $6x^2 - 11x + 4$, and it is therefore the H. C. D. In such a case, however, it is generally customary to remove the factor -2 to make the first term of each divisor positive. It is important to notice in this connection that the highest common divisor when found is a certain quantity, or *its opposite*. In the preceding examples, the H. C. D. is $2x - 3$ or $-2x + 3$, $2x - 1$ or $-2x + 1$, $3x - 4$ or $-3x + 4$.

Show that a divisor of a quantity is a divisor of any number of times that quantity.

Show that a common divisor of two quantities is a divisor of their sum and also of their difference.

Show that the highest common divisor of two or more quantities is the product of their common factors.

4.
$$
\begin{array}{l|l}
6x^3 - x^2 - 8x + 4 & 3x^2 + x - 2 \\
6x^3 + 2x^2 - 4x & 2x - 1 \\ \hline
\end{array}
$$
$$
\begin{array}{r}
-3x^2 - 4x + 4 \\
-3x^2 - x + 2 \\ \hline
-1)\underline{-3x + 2}
\end{array}
$$
$$
\begin{array}{r}
3x - 2)3x^2 + x - 2(x + 1 \\
\underline{3x^2 - 2x\phantom{{}-2}} \\
3x - 2 \\
\underline{3x - 2}
\end{array}
$$

$\therefore$ the H. C. D. is $3x - 2$.

NOTE 3. — Notice the removal of the factor -1 in the remainder $-3x + 2$ in the fourth example.

5. $6x^5y - 42x^4y + 78x^3y - 90x^2y = 6x^2y(x^3 - 7x^2 + 13x - 15)$
$4x^4y^2 - 24x^3y^2 + 4x^2y^2 + 80xy^2 = 4xy^2(x^3 - 6x^2 + x + 20)$

These polynomials have the common factor $2xy$.

$$
\begin{array}{l|l}
x^3 - 7x^2 + 13x - 15 & x^3 - 6x^2 + x + 20 \\
x^3 - 6x^2 + x + 20 & 1 \\ \hline
\end{array}
$$
$$
\begin{array}{r}
-1)\underline{-x^2 + 12x - 35} \\
x^2 - 12x + 35)x^3 - 6x^2 + x + 20(x + 6 \\
\underline{x^3 - 12x^2 + 35x\phantom{{}+20}} \\
6x^2 - 34x + 20 \\
\underline{6x^2 - 72x + 210} \\
38)\underline{38x - 190} \\
x - 5)x^2 - 12x + 35(x - 7 \\
\underline{x^2 - 5x\phantom{{}+35}} \\
-7x + 35 \\
\underline{-7x + 35}
\end{array}
$$

$\therefore$ the H. C. D. is $2xy(x - 5)$, or $2x^2y - 10xy$.

Note 4.—The given polynomials in the fifth example contain each a simple factor. These simple factors are first removed, $6x^2y$ from the first and $4xy^3$ from the second. The factor $2xy$, being common to both, must be reserved as a factor of the H. C. D. Proceeding with the compound factors, as in the previous examples, their H. C. D. is found to be $x-5$. Therefore the H. C. D. sought is $2xy(x-5)$.

$$
\begin{array}{rl}
6.\quad 2x^5 - x^3 - x^2 + 3x - 4 & \big|\ \underline{3x^4 - 4x^3 + 6x^2 - 3x + 2} \\
3 & \big|\ 2x + 8 \\
\hline
6x^5 - 3x^3 - 3x^2 + 9x - 12 & \\
6x^5 - 8x^4 + 12x^3 - 6x^2 + 4x & \\
\hline
8x^4 - 15x^3 + 3x^2 + 5x - 12 & \\
3 & \\
\hline
24x^4 - 45x^3 + 9x^2 + 15x - 36 & \\
24x^4 - 32x^3 + 48x^3 - 24x + 16 & \\
\hline
-13\underline{)-13x^3 - 39x^2 + 39x - 52} & \\
x^3 + 3x^2 - 3x + 4 &
\end{array}
$$

$$
\begin{array}{rl}
3x^4 - 4x^3 + 6x^2 - 3x + 2 & \big|\ \underline{x^3 + 3x^2 - 3x + 4} \\
3x^4 + 9x^3 - 9x^2 + 12x & \big|\ 3x - 13 \\
\hline
-13x^3 + 15x^2 - 15x + 2 & \\
-13x^2 - 39x^2 + 39x - 52 & \\
\hline
54\underline{)54x^2 - 54x + 54} & \\
x^2 - x + 1 &)\,x^3 + 3x^2 - 3x + 4\,(x + 4 \\
& \underline{x^3 - x^2 + x} \\
& 4x^2 - 4x + 4 \\
& \underline{4x^2 - 4x + 4}
\end{array}
$$

$\therefore$ the H. C. D. is $x^2 - x + 1$.

Note 5.—It often occurs, as in the sixth example, that the first term of the dividend is not exactly divisible by the first term of the

divisor. Any dividend may be multiplied to make the first term of it exactly divisible by the divisor, for the introduction of a factor into one of the polynomials and not into the other can not affect the H. C. D.

148. To find the highest common divisor by continued division.

RULE.—*If the polynomials contain a common monomial factor, remove such factor and reserve it as a factor of the highest common divisor.*

If either of the polynomials contains a monomial factor that is not found in the other, remove such factor and reject it.

Arrange the remaining polynomials with reference to the descending powers of the same letter, and divide the polynomial of the higher degree by the polynomial of the lower degree, continuing the division until there is no remainder, or until the first term of the remainder is of a lower degree than the first term of the divisor.

Divide the final remainder, if any, by the preceding divisor, and so continue until there is no remainder. The last divisor multiplied by the common monomial factor removed in the beginning, if any, will be the highest common divisor.

If three polynomials are given, find the highest common divisor of two of them; then take that highest common divisor and the remaining polynomial, and find their highest common divisor. The last result will be the highest common divisor of all.

Show that a common divisor of two quantities is a divisor of any multiple of either of them.

Show that the highest common divisor of two quantities is not affected by multiplying or dividing either of them by a quantity that is not a factor of the other.

If two polynomials each contain a monomial factor, how much of those monomial factors, if any, should be reserved as a factor of the highest common divisor?

LOWEST COMMON MULTIPLE.

149. A Multiple of a quantity is a quantity that is exactly divisible by that quantity.

150. A Common Multiple of two or more quantities is a quantity that is exactly divisible by each of them.

151. The Lowest Common Multiple of two or more quantities is the quantity of the lowest degree that is exactly divisible by each of them.

152. PRINCIPLES. — 1. *Every multiple of a quantity must contain all the factors of that quantity.*

2. *A common multiple of two or more quantities must contain all the factors of each of them, and it may contain others.*

3. *The lowest common multiple of two or more quantities must contain all the factors of each of them, but no others.*

4. *If two quantities are prime to each other, their lowest common multiple is their product.*

153. To find the lowest common multiple of monomials.

RULE. — *To the least common multiple of the numerical coefficients, annex all the letters of the several monomials, giving each letter the highest exponent that it has in any of the monomials.*

154. To find the lowest common multiple of quantities that are easily factored.

RULE. — *Resolve the quantities into their prime factors, and find the product of all the different factors, taking each factor as many times as it is found in any of the quantities.*

EXAMPLE.

$$x^2+7x+12=(x+3)(x+4)$$
$$x^2+8x+16=(x+4)(x+4)$$
$$x^2-4x-32=(x+4)(x-8)$$
$$\therefore \text{ the L. C. M. is } (x+3)(x+4)(x+4)(x-8).$$

To the Teacher. — As lowest common multiple is used in addition and subtraction of fractions and in clearing equations of fractions, students should be trained to determine quickly by inspection the lowest common multiple in simple examples. Show that if the lowest common multiple of two or more quantities be divided by one of its factors, the quotient will be the other factor, or the product of the other factors of the lowest common multiple. After students have determined the lowest common multiple in the given examples, with the factors before them, require them to give the quotient of the lowest common multiple divided by each of the quantities in the example. This will be valuable preparatory drill for subsequent work in fractions and equations.

155. To find the lowest common multiple of quantities that are not easily factored.

Rule. — *Find the highest common divisor of the quantities, divide one of them by the highest common divisor, and multiply the other quantity by the quotient.*

EXAMPLE.

$$x^3 - 4x^2 - 6x + 5$$
$$x^3 - 5x^2 - 3x + 15$$

The H. C. D. is $x - 5$

$$(x^3 - 5x^2 - 3x + 15) \div (x - 5) = x^2 - 3$$

$\therefore$ the L. C. M. is $(x^3 - 4x^2 - 6x + 5)(x^2 - 3)$.

FRACTIONS.

156. **A Fraction** is one or more of the equal parts of a unit.

157. To express a fraction, two quantities are necessary; one to express the number of equal parts into which the number or thing is divided, and the other to express how many of the parts are taken. They are written, one above, and the other below, the same horizontal line.

158. **The Denominator** is the number or quantity written below the line. It shows into how many equal parts the number or thing is divided.

159. The Numerator is the number or quantity written above the line. It shows how many equal parts are taken.

160. The Terms of a fraction are the numerator and denominator.

161. An Entire Quantity is a quantity no part of which is a fraction.

162. A Mixed Quantity is an entire quantity and fraction written together.

163. A Proper Fraction is a fraction which can not be reduced to an entire or mixed quantity.

164. An Improper Fraction is a fraction which can be reduced to an entire or mixed quantity.

165. The Value of a Fraction is the quotient of the numerator divided by the denominator.

166. The Sign of a Fraction is the sign written before the fraction. Fractions, like entire quantities, are positive or negative. Each term of the fraction is also either positive or negative. There are therefore three signs belonging to every fraction; the sign of the numerator, the sign of the denominator, and the sign of the fraction.

167. If the terms of a fraction have like signs, and the sign of the fraction is positive, the value of the fraction is positive.

168. If the terms of a fraction have unlike signs, and the sign of the fraction is negative, the value of the fraction is positive.

169. If the terms of a fraction have like signs, and the sign of the fraction is negative, the value of the fraction is negative.

170. If the terms of a fraction have unlike signs, and the sign of the fraction is positive, the value of the fraction is negative.

171. PRINCIPLES.—1. *Multiplying the numerator by any quantity multiplies the fraction by that quantity.*

2. *Dividing the numerator by any quantity divides the fraction by that quantity.*

3. *Multiplying the denominator by any quantity divides the fraction by that quantity.*

4. *Dividing the denominator by any quantity multiplies the fraction by that quantity.*

5. *Multiplying or dividing both terms of a fraction by the same quantity does not change the value of the fraction.*

6. *Changing any two signs of a fraction does not change the value of the fraction.*

7. *Changing one or all of the signs of a fraction changes the value of the fraction from positive to negative, or from negative to positive.*

NOTE.—It must be remembered that if the numerator is a compound expression, the sign of the numerator is changed by changing the sign of every term of it. The same is true of the denominator.

172. Reduction of Fractions is the process of changing their form without changing their value.

173. A fraction is reduced to its lowest terms when the numerator and denominator are prime to each other.

174. To reduce a fraction to its lowest terms.

RULE.—*Cancel all factors common to numerator and denominator.*

175. To reduce an improper fraction to an entire or mixed quantity.

RULE.—*Divide the numerator by the denominator.*

NOTE 1. — If the division is exact, the fraction reduces to an entire quantity.

NOTE 2. — If there is a remainder and the sign of its first term is plus, write the remainder over the divisor and annex it to the entire part, connecting them with the plus sign.

NOTE 3. — If the sign of the first term of the remainder is minus, change the sign of each term of the remainder, write it over the divisor, and annex it to the entire part, connecting them with the minus sign.

EXAMPLES.

1. $\dfrac{x^4-1}{x-1}$ $\qquad \dfrac{x^4-1}{x-1}=x^3+x^2+x+1.$

2. $\dfrac{x^3+a}{x+3}$

$$\begin{array}{l|l}
x^3+a & x+3 \\ \cline{2-2}
x^3+3x^2 & x^2-3x+9 \\ \cline{1-1}
\quad -3x^2+a & \\
\quad -3x^2-9x & \\ \cline{1-1}
\qquad\qquad 9x+a & \\
\qquad\qquad 9x+27 & \\ \cline{1-1}
\qquad\qquad\ \ a-27 &
\end{array}$$

$$\therefore\ \frac{x^3+a}{x+3}=x^2-3x+9+\frac{a-27}{x+3}.$$

3. $\dfrac{6x^3-4x^2-12x+10}{2x^2-3}$

$$\begin{array}{l|l}
6x^3-4x^2-12x+10 & 2x^2-3 \\ \cline{2-2}
6x^3 \qquad\ \ -\ 9x & 3x-2 \\ \cline{1-1}
\quad -4x^2-\ 3x+10 & \\
\quad -4x^2 \qquad\ +\ 6 & \\ \cline{1-1}
\qquad\qquad -\ 3x+\ 4 &
\end{array}$$

$$\therefore\ \frac{6x^3-4x^2-12x+10}{2x^2-3}=3x-2-\frac{3x-4}{2x^2-3}.$$

176. To reduce a mixed quantity to an improper fraction.

RULE. — *Multiply the entire part of the denominator of the fraction; to the product add the numerator of the fraction, if*

the sign before the fraction is plus, and subtract the numerator, if the sign before the fraction is minus; and write the result over the denominator.

EXAMPLES.

1. $x - 4 + \dfrac{x^2+8}{x+3}$

$$\begin{array}{rr} (x-4)(x+3) = & x^2 - x - 12 \\ \text{Adding} & x^2 \quad\; + 8 \\ \hline & 2x^2 - x - 4 \end{array}$$

$$\therefore x - 4 + \frac{x^2+8}{x+3} = \frac{2x^2-x-4}{x+3}.$$

2. $a - b - \dfrac{a^2+b^2}{a-b}$

$$\begin{array}{rr} (a-b)(a-b) = & a^2 - 2ab + b^2 \\ \text{Subtracting} & a^2 \qquad\quad + b^2 \\ \hline & -2ab \end{array}$$

$$\therefore a - b - \frac{a^2+b^2}{a-b} = \frac{-2ab}{a-b} \text{ or } -\frac{2ab}{a-b}.$$

177. To reduce fractions to their lowest common denominator.

RULE. — *Find the lowest common multiple of the denominators for the required denominator.*

Divide this denominator by the denominator of each fraction, and multiply the numerator by the quotient.

TO THE TEACHER. — Train students to express the denominator of each fraction and the lowest common denominator in their factors. Then show that the quotient of the lowest common denominator divided by each denominator is all the factors of the lowest common denominator that are not found in the denominator of the fraction. This practice should also be observed in addition and subtraction of fractions. Students will save much time and labor if, in all work in algebra, they never multiply factors together until it is found to be necessary.

ADDITION AND SUBTRACTION OF FRACTIONS.

178. **Like Fractions** are fractions that express like fractional units.

179. To express like fractional units, fractions must have a common denominator.

180. PRINCIPLE. — *Only like fractions can be added.*

181. RULE FOR ADDITION. — *If necessary, reduce the fractions to their lowest common denominator, add the numerators, and write the sum over the common denominator.*

NOTE. — If the sum of the numerators is zero, the value of the fraction is zero.

182. PRINCIPLE. — *A fraction can be subtracted only from a like fraction.*

183. RULE FOR SUBTRACTION. — *If necessary, reduce the fractions to their lowest common denominator, subtract the numerator of the subtrahend from the numerator of the minuend, and write the remainder over the common denominator.*

184. To combine several fractions which are connected by the signs of addition and subtraction.

RULE. — *Reduce the fractions to their lowest common denominator, write the several numerators for adding, writing those to be subtracted with their signs changed, add the numerators thus written, and place the sum over the common denominator.*

TO THE TEACHER. — As addition and subtraction of fractions are so nearly alike, examples in the two subjects are given together in the same lesson.

185. In some cases, it is desirable to change the form of a fraction before adding or subtracting, as in the following example:

$$\frac{3}{a-x}+\frac{5}{a+x}+\frac{2x+a}{x^2-a^2}$$

As the fractions are given, the lowest common denominator is the product of the denominators. If the third denominator were $a^2 - x^2$, it would be the lowest common denominator. Dividing both terms of a fraction by the same quantity does not change the value of the fraction, therefore both terms of the third fraction may be divided by -1. The denominator will then be $-x^2 + a^2$, or $a^2 - x^2$, and the numerator will be $-2x - a$. Changing any two signs of a fraction, as the sign of the numerator and the sign before the fraction, does not change the value of the fraction, hence the numerator may be changed to $2x + a$, leaving it as it was at first, and the sign before the fraction changed from plus to minus. Therefore, to make the desired change in the form of the fraction, change the signs in the denominator and the sign before the fraction.

Sometimes the denominator of a fraction is expressed in its factors, as in the following:

$$\frac{1}{(x-1)(x-2)} + \frac{1}{(2-x)(x-3)} + \frac{1}{(1-x)(3-x)}$$

It is desirable to change the first factor in the second denominator to $x - 2$, and the factors in the third denominator to $x - 1$ and $x - 3$. In dividing any number, that is expressed by its factors, by 2, for example, only one factor must be divided by 2. Dividing both terms of the second fraction by -1, remembering that in dividing the denominator, only the one factor, $2 - x$, is divided, the numerator becomes -1 and the denominator $(x-2)(x-3)$. The sign of the numerator and the sign before the fraction may then be changed as in the first example. Dividing both terms of the third fraction by -1 the first time, the numerator becomes -1 and the denominator $(x-1)(3-x)$. Dividing both terms by -1 again, the numerator becomes 1 and the denominator $(x-1)(x-3)$.

From the above the following principles are deduced:

1. *If the signs of one factor in the denominator are changed, the sign before the fraction must be changed.*

2. *If the signs of two factors in the denominator are changed, the sign before the fraction remains the same.*

Multiplication of Fractions.

186. In multiplication of fractions, one or more of the factors are fractions.

187. RULE. — *Express all factors in the fractional form. Cancel all factors common to numerators and denominators, and multiply the remaining factors in the numerators together for the numerator of the product, and the remaining factors in the denominators together for the denominator of the product.*

DIVISION OF FRACTIONS.

188. In division of fractions, either the divisor or dividend is a fraction. Both may be.

189. RULE. — *Express both dividend and divisor in the fractional form, invert the divisor, and proceed as in multiplication of fractions.*

190. **A Complex Fraction** is a fraction which has a fraction in its numerator, or in its denominator, or in both.

191. **To reduce a complex fraction to a simple one.**

RULE. — *Perform the indicated operations.*

SIMPLE EQUATIONS.

192. **An Equation** is the expression of the equality of two numbers or quantities.

NOTE. — It is hoped that students using this book will pronounce this word *e qua shun*, not *e qua zion.*

193. **The Members** of an equation are the quantities at the right and left of the sign of equality.

194. **The First Member** of an equation is the quantity at the left of the sign of equality.

195. **The Second Member** of an equation is the quantity at the right of the sign of equality.

196. Equations are distinguished as *Identical Equations*, or *Identities*, and *Equations of Condition.*

197. **An Identical Equation, or Identity,** is an equation in which one member is exactly the same as the other, or in which one member is a development of the other. In an identity, the two members are equal, whatever number each letter represents.

The *sign of identity* is $\equiv$. It is read *is identical to*. Identities may also be expressed with the sign of equality.

$$ax - c \equiv ax - c$$
$$(x + 2)(x - 9) \equiv x^2 - 7x - 18$$
$$(a + x)(b - y) \equiv ab - ay + bx - xy$$

Examples in addition, subtraction, multiplication, division, fractions, and factoring, when expressed in the form of equations, are examples of identities.

198. **An Equation of Condition** is an equation the two members of which are equal only when each letter represents some particular value. See Art. 6, page 13.

The term *equation*, as commonly used in algebra, means an equation of condition.

199. **A Numerical Equation** is an equation in which all the known quantities are represented by figures.

$8x - 5 = 6x + 10 - x \qquad 5x + 3x - 2x = 3x + 24$

200. **A Literal Equation** is an equation in which some or all of the known quantities are represented by letters.

$2x + a = b - 4 \qquad\qquad 2x + b = a + x$

201. **The Degree** of an equation is determined by the largest number of factors of the unknown quantities in any term.

$5x + 7 = 2x - a$ is an equation of the first degree.

$4x - 3 = xy - b$ is an equation of the second degree.

202. **A Simple Equation** is an equation of the first degree.

203. An Axiom is a self-evident truth.

204. Axioms.—1. *Quantities that are equal to the same quantity are equal to each other.*

2. *If the same quantity or equal quantities be added to equal quantities, the sums will be equal.*

3. *If the same quantity or equal quantities be subtracted from equal quantities, the remainders will be equal.*

4. *If equal quantities be multiplied by the same quantity or equal quantities, the products will be equal.*

5. *If equal quantities be divided by the same quantity or equal quantities, the quotients will be equal.*

6. *Equal powers of equal quantities are equal.*

7. *Equal roots of equal quantities are equal.*

205. Transformation of equations is the process of changing their form without destroying the equality of the members.

Every transformation of an equation involves one or more of the foregoing axioms, and it is of the utmost importance that students should be able to show what axiom is involved in each change.

Transposition.

206. Transposition is the process of changing a term from one member of an equation to the other.

207. Principles.—1. *If the same quantity be added to equal quantities, the sums will be equal.*

2. *If the same quantity be subtracted from equal quantities, the remainders will be equal.*

208. To transpose a term from one member of an equation to the other.

Rule.—*Write the term in the opposite member with its sign changed.*

Clearing Equations of Fractions.

209. Principles. — 1. *If a fraction be multiplied by its denominator or any multiple of its denominator, the product will be an entire quantity.*

2. *If several fractions be multiplied by a common multiple of their denominators, the products will be entire quantities.*

3. *If equal quantities be multiplied by the same quantity, the products will be equal.*

210. Rule. — *Multiply both members of the equation by the lowest common multiple of the denominators.*

EXAMPLES.

1. $$\frac{x}{2}-10+\frac{x}{4}+3-\frac{x}{3}=\frac{x}{6}-5$$

$$6x-120+3x+36-4x=2x-60$$

2. $$\frac{4}{x-1}+\frac{x-1}{x+1}=\frac{5}{x+1}+\frac{x+1}{x-1}$$

$$4x+4+x^2-2x+1=5x-5+x^2+2x+1$$

211. If a fraction is preceded by the minus sign, the sign of each term of the numerator must be changed when the equation is cleared of fractions.

EXAMPLES.

1. $$\frac{x}{2}-\frac{x-8}{6}-\frac{x+12}{3}=\frac{2-x}{8}$$

$$12x-4x+32-8x-96=6-3x$$

2. $$\frac{3}{a+x}-\frac{a+x}{a-x}-\frac{a-x}{a+x}=2-\frac{4}{a-x}$$

$$3a-3x-a^2-2ax-x^2-a^2+2ax-x^2=2a^2-2x^2-4a-4x$$

Solution of Simple Equations.

212. The Root of an equation is the value of the unknown quantity.

213. The Solution of an equation is the process of finding the root of the equation.

214. To solve a simple equation.

Rule. — *If necessary, clear the equation of fractions.*

Transpose all the terms containing the unknown quantity to the first member of the equation, and all the known quantities to the second member of the equation.

Unite all the terms containing the unknown quantity into one term, and unite similar terms in the second member.

Divide both members of the equation by the coefficient of the unknown quantity.

EXAMPLES.

1. $$\frac{x}{2}+\frac{x}{3}-5=\frac{x}{4}-\frac{x-6}{6}$$

$$6x+4x-60=3x-2x+12$$

$$6x+4x-3x+2x=60+12$$

$$9x=72$$

$$x=8$$

2. $$ax-b=bx-c$$

$$ax-bx=b-c$$

$$(a-b)x=b-c$$

$$x=\frac{b-c}{a-b}$$

3. $\frac{x}{a}+\frac{x}{b}+3=\frac{3}{a}-\frac{x-2}{b}$

$$bx+ax+3ab=3b-ax+2a$$
$$bx+ax+ax=3b+2a-3ab$$
$$(b+2a)x=3b+2a-3ab$$
$$x=\frac{3b+2a-3ab}{b+2a}$$

NOTE 1. — When the same quantity is found in both members of the equation *with the same sign*, it may be canceled from both.

NOTE 2. — The signs of all the terms of an equation may be changed without destroying the equality.

NOTE 3. — It often happens in the solution of equations that when the terms containing the unknown quantity are united, the coefficient is minus. If the first member is $-4x$, divide both members by -4; if $-6x$, divide by -6; if $-x$, divide by -1; if $-ax$, divide by $-a$; etc.

SPECIAL SOLUTIONS.

215. In clearing equations of fractions, if some of the denominators are monomials and some of them polynomials, it is often advantageous to remove the monomial denominators first and then the polynomial denominators.

EXAMPLES.

1. $\frac{2x-9}{5}+\frac{2x+6}{3x-1}=\frac{4x-8}{10}$

$$4x-18+\frac{20x+60}{3x-1}=4x-8$$
$$\frac{20x+60}{3x-1}=10$$
$$20x+60=30x-10$$
$$-10x=-70$$
$$x=7$$

2. $$\frac{6x-3}{15}-\frac{3x-4}{2x+4}=\frac{2x-6}{5}$$

$$6x-3-\frac{45x-60}{2x+4}=6x-18$$

$$-\frac{45x-60}{2x+4}=-15$$

$$\frac{45x-60}{2x+4}=15$$

$$45x-60=30x+60$$

$$15x=120$$

$$x=8$$

216. Sometimes it is best to combine fractions before clearing of fractions.

EXAMPLE.

$$\frac{1}{a-c}+\frac{a-c}{x}=\frac{1}{a+c}+\frac{a+c}{x}$$

$$\frac{a-c}{x}-\frac{a+c}{x}=\frac{1}{a+c}-\frac{1}{a-c}$$

$$\frac{a-c-a-c}{x}=\frac{a-c-a-c}{a^2-c^2}$$

$$\frac{-2c}{x}=\frac{-2c}{a^2-c^2}$$

$$\frac{1}{x}=\frac{1}{a^2-c^2}$$

$$x=a^2-c^2$$

NOTE.—From the last solution it appears that if two fractions having the same numerator are equal, the denominators are equal.

217. The verification of the result is the process of proving that the root satisfies the equation.

218. To verify the root of an equation.

RULE. — *Substitute the value of the unknown quantity for its symbol in the original equation. If it renders the two members identical, the value of the unknown quantity is correct.*

EXAMPLES.

1. $$3x - 8 + 2x + 3 = 2x + 10$$
$$3x + 2x - 2x = 10 + 8 - 3$$
$$3x = 15$$
$$x = 5$$

VERIFICATION.

$$15 - 8 + 10 + 3 = 10 + 10$$
$$20 = 20$$

2. $$\frac{x}{2} - \frac{x-4}{5} = \frac{x}{4} + \frac{x-8}{8}$$
$$20x - 8x + 32 = 10x + 5x - 40$$
$$20x - 8x - 10x - 5x = -40 - 32$$
$$-3x = -72$$
$$x = 24$$

VERIFICATION.

$$\tfrac{24}{2} - \tfrac{20}{5} = \tfrac{24}{4} + \tfrac{16}{8}$$
$$12 - 4 = 6 + 2$$
$$8 = 8$$

PROBLEMS.

219. **A Problem** is a question requiring solution.

220. **The Solution** of a problem is the process of finding the unknown number or quantity.

221. The solution of a problem involves two operations, the *Statement*, and the *Solution of the Equation*.

222. The Statement of a problem is the expression of the conditions of the problem in algebraic symbols in the form of one or more equations.

223. In the solution of problems in arithmetic, the work proceeds from what is given toward the desired result; while in algebra, the problem is conceived as solved, and the work proceeds as though the result were to be verified. In this work, a letter everywhere takes the place of the unknown number. An equation is formed, and the solution of this equation gives the number represented by the letter, or the number sought.

224. The statement of problems is the more difficult part of the solution. Since the conditions of problems in algebra are not the same, no rule can be given for the statement of them. The following directions will be helpful to the student.

1. *Let x, or some multiple of x, represent the unknown quantity, or number.*
2. *Express in algebraic language each statement of the problem.*
3. *Determine from the conditions of the problem what quantities are equal.*
4. *Find algebraic expressions for these equal quantities.*
5. *Express the equality of these expressions.*
6. *Solve the equation thus formed.*

Additional Suggestions.

225. In the statement of problems, remember that the whole of a number or thing is equal to the sum of its parts.

226. If a problem states that certain quantities are equal, find algebraic expressions for the two quantities and place them equal.

227. To avoid fractions in the statement of problems, it is often advisable to let some multiple of x represent the unknown quantity.

EXAMPLE.

A has $\frac{1}{2}$ as many sheep as B, B has $\frac{2}{3}$ as many as C, and together they have 480. How many has each?

Let $12x =$ the number C has;
then $8x =$ the number B has,
and $4x =$ the number A has.

Since all have 480 sheep,

$$12x + 8x + 4x = 480$$
$$24x = 480$$
$$x = 20$$
$$4x = 80, \text{ the number A has,}$$
$$8x = 160, \text{ the number B has,}$$
$$12x = 240, \text{ the number C has.}$$

228. In many problems, the equation is formed by getting two different expressions to represent the same thing and placing them equal to each other. See problem 69, page 114; also problem 88, page 116.

229. If the statement of a problem gives a proportion, an equation may be formed by placing the product of the means equal to the product of the extremes.

When a problem states that two quantities are to each other as two numbers, as 3 to 5, for example, it means that the first quantity is 3 times a measuring unit, and the second quantity 5 times the same measuring unit. In such problems, it is convenient to represent the measuring unit by x. Then $3x$ and $5x$ will represent the quantities. In such a solution, the unknown quantities sought will be certain multiples of the value of x.

230. If a problem states that one quantity is a certain number of times another, find algebraic expressions for the

two quantities, divide the greater quantity by the ratio and place the result equal to the smaller quantity; or multiply the smaller quantity by the ratio, and place the result equal to the greater quantity. See problem 8, page 108; also problem 53, page 112.

231. If a problem states that one quantity is greater or less than another and gives the difference, find algebraic expressions for the two quantities, subtract the difference from the greater quantity, and place the result equal to the smaller quantity; or add the difference to the smaller quantity, and place the result equal to the greater quantity. See problems 11 and 17, page 109.

232. The two members of an equation must represent like numbers. An expression representing dollars must not be placed equal to an expression representing cents; an expression representing time must not be placed equal to an expression representing distance; an expression representing yards must not be placed equal to an expression representing feet or inches.

233. Every part of the statement should be definite, and it should be clearly indicated what each expression represents. x must always represent some number. It must not equal money, but a *number* of dollars or cents; it must not equal time, but a *number* of years, days, or hours; it must not equal weight, but a *number* of pounds or ounces; it must not equal distance, but a *number* of miles, rods, yards, feet, or inches.

234. Many problems given in algebra involve the principles of interest or percentage. Six per cent of any number is $\frac{6}{100}$ of it. In arithmetic, any per cent of a number is usually found by multiplying the number by the rate

expressed decimally; but in algebra, it is customary to express the rate and the product obtained in the form of a common fraction. Thus, 5 per cent of x is $\frac{5}{100}$ of x, or $\frac{5x}{100}$; x per cent of 800 is $\frac{x}{100}$ of 800, or $\frac{800x}{100}$; etc.

235. Problems are often given in algebra in which it is required to find the time it will take two or more persons to do a piece of work. These problems generally state the time required for each to do the work. In all such problems, the method of solution is similar. Let x represent the number of units of time required. Find the fraction of the work that each can do in one unit of time, and then the fraction that he can do in x units of time. By Art. 225, the sum of the fractions that they can do in x units of time must equal the whole work, or 1.

Such problems may also be solved by finding the fraction that all can do in one unit of time, and the fraction that each can do in one unit of time, and placing the sum of the fractions that each can do in one unit of time equal to the fraction that all can do in one unit of time.

236. Problems are often given which relate to the movement of the hands of a clock. These are stated in various ways, but in all of them it is required to find at what time or times the hands will be in certain relative positions. The most common requirement in these problems is to find at what time between certain hours the hands will be together, or at what time the hands will be a certain number of minute-spaces apart. In the solution of such problems, let x represent the number of minute-spaces passed over by the minute-hand before the required conditions are fulfilled. It is evident that the number of minute-spaces passed over by the minute-hand determines the number of minutes after the given hour. Now, since the minute-hand goes 12 times

as fast as the hour-hand, $\frac{x}{12}$ will represent the number of minute-spaces passed over by the hour-hand in the same time. At the beginning of any given hour, the minute-hand is at 12, and at every hour except 12 the hour-hand is ahead of the minute-hand. In forming the equation, it is necessary to determine whether the condition is fulfilled before the minute-hand passes the hour-hand, when the hands are together, or when the minute-hand is in advance of the hour-hand. In the first case, x is equal to the number of spaces the hour-hand is in advance of the minute-hand, plus $\frac{x}{12}$, minus the number of spaces the hour-hand is ahead of the minute-hand when the required condition is fulfilled. In the second case, x is equal to the number of spaces the hour-hand is in advance of the minute-hand, plus $\frac{x}{12}$. In the third case, x is equal to the number of spaces the hour-hand is in advance of the minute-hand, plus $\frac{x}{12}$, plus the number of spaces the minute-hand is ahead of the hour-hand when the required condition is fulfilled.

To the Teacher.—A careful study of the following questions will prepare students for the statement of problems.

1. If a man has x sheep in one field and y in another, how many has he in both fields?

2. If you are x years old to-day, how old will you be in four years?

3. The difference of two numbers is 18. If the smaller number is x, what is the larger number?

4. A man bought a coat for x dollars, and sold it at a profit of 4 dollars. How much did he receive for it?

5. A man sold a boat for x dollars and lost y dollars. How much did he pay for the boat?

6. The sum of two numbers is y. If the larger number is x, what is the smaller number?

7. The difference of two numbers is 18. If the larger number is x, what is the smaller number?

8. If a man is x years old to-day, how old was he fourteen years ago?

9. A man bought a horse for y dollars, and sold it at a loss of x dollars. How much did he receive for the horse?

10. A man sold a wagon for 80 dollars and gained b dollars. How much did he pay for the wagon?

11. A boy bought x apples at a cents apiece, and sold them at b cents apiece. If he gained, what was his gain?

12. A man bought x sheep at a dollars a head, and y sheep at b dollars a head. How much did they all cost?

13. If you are x years old, how old is your father, who is three times as old?

14. If a boy has x half-dollars, how many cents has he?

15. At x dollars a head, how much will 125 sheep cost?

16. A has x sheep, and B has y. How many would C have, if he had half as many as A and B together?

17. If you have x quarters and y dimes, how many cents have you?

18. If a man can do a piece of work in x days, what part of it can he do in one day?

19. If you have x half-dollars and y quarters, how many dollars have you?

20. A boy received a cents from his father, earned b cents, and spent 8 cents for candy. How much had he left?

21. A man paid x dollars for three pair of shoes. How much did each pair cost?

22. A farmer received y dollars for sheep, which he sold at x dollars a head. How many did he sell?

23. A merchant sold x yards of silk for \$45. At what price per yard did he sell it?

24. If a yards of silk cost x dollars, how much will twenty-four yards cost?

25. A man sold a bushels of apples at b cents a peck and x bushels of pears at y cents a peck. How much did he receive for both?

26. The sum of the ages of three boys is $4x$ years. What will be the sum of their ages in six years?

27. If 8 books cost x dollars, how many books can be bought for a dollars at the same price?

28. If a man can do a piece of work in 8 days, what part of it can he do in x days?

29. What is the interest of a dollars for four years at x per cent per annum?

30. What is the interest of a dollars for x months at five per cent per annum?

31. What is the interest of a dollars for three years and eight months at x per cent per annum?

32. If x is the smallest of four consecutive numbers, what are the other numbers?

33. If A has x sheep, how many has B, who has 18 less than $\frac{5}{7}$ as many as A?

34. What is the number in which there are x hundreds, y tens, and z units?

35. What is the area of a rectangle whose length is x feet and whose width is y feet?

36. What is the area of a rectangle whose length is x feet and whose width is y inches?

37. If the dividend is a, the divisor b, and the remainder c, what is the quotient?

38. A man bought x acres of land at a dollars per acre and sold it at a profit of b dollars an acre. How much did he get for it?

39. A speculator bought a horses at y dollars a head and sold them at a loss of x dollars a head. How much did he get for them?

40. A grocer bought a pounds of tea at x cents a pound and sold it at y cents a pound. If y is greater than x, did he gain or lose? How much?

Statement and Solution of Problems.

1. One-half of a certain number increased by two-fifths of the same number equals 81. What is the number?

Let $x =$ the number;

then $\frac{x}{2} =$ one-half of the number,

and $\frac{2x}{5} =$ two-fifths of the number.

Since the *sum* of the two parts is 81,

$$\frac{x}{2} + \frac{2x}{5} = 81$$

$$5x + 4x = 810$$

$$x = 90, \text{ the required number.}$$

2. Three boys together have 350 marbles. Frank has 20 more than John, and Harry has 25 more than Frank. How many has each?

Let $x =$ the number John has;

then $x + 20 =$ the number Frank has,

and $x + 45 =$ the number Harry has.

Since they *all* have 350 marbles,

$$x + x + 20 + x + 45 = 350$$
$$3x = 285$$
$$x = 95, \text{ the number John has,}$$
$$95 + 20 = 115, \text{ the number Frank has,}$$
$$95 + 45 = 140, \text{ the number Harry has.}$$

3. The difference between two numbers is 37, and if 131 be added to the larger number, the result will be three times the smaller number. Find the numbers.

Let $x =$ the larger number;
then $x - 37 =$ the smaller number.

Since the larger number, increased by 131, is equal to three times the smaller,

$$x + 131 = 3x - 111$$
$$-2x = -242$$
$$x = 121, \text{ the larger number,}$$
$$121 - 37 = 84, \text{ the smaller number.}$$

ANOTHER SOLUTION.

Let $x =$ the smaller number;
then $x + 37 =$ the larger number.

Since the larger number, increased by 131, is equal to three times the smaller,

$$x + 37 + 131 = 3x$$
$$-2x = -168$$
$$x = 84, \text{ the smaller number,}$$
$$84 + 37 = 121, \text{ the larger number.}$$

4. Divide 59 into two parts such that four times the smaller part shall exceed twice the larger part by 26.

Let $x =$ the smaller number;
then $59 - x =$ the larger number.

Since four times the smaller number exceeds twice the larger by 26,

$$4x - 26 = 118 - 2x$$
$$6x = 144$$
$$x = 24, \text{ the smaller number,}$$
$$59 - 24 = 35, \text{ the larger number.}$$

ANOTHER SOLUTION.

Let $x =$ the larger number;

then $59 - x =$ the smaller number.

Since four times the smaller number exceeds twice the larger by 26,

$$4(59 - x) - 26 = 2x$$

$$236 - 4x - 26 = 2x$$

$$-6x = -210$$

$x = 35$, the larger number,

$59 - 35 = 24$, the smaller number.

5. A man sold some sheep at $3 a head and three times as many at $4 a head, receiving for all $375. How many did he sell?

Let $x =$ the number sold at $3 a head;

then $3x =$ the number sold at $4 a head.

$3x =$ the number of dollars received for the first,

and $12x =$ the number of dollars received for the second.

Since he received for all $375,

$$3x + 12x = 375$$

$$15x = 375$$

$x = 25$, the number sold at $3 a head,

$3x = 75$, the number sold at $4 a head,

$75 + 25 = 100$, the whole number sold.

6. The sum of a third and fourth of a number exceeds the difference between a half and third of the number by 50. Find the number.

Let $x =$ the number;

then $\frac{x}{3} =$ one-third of the number,

and $\frac{x}{4} =$ one-fourth of the number,

and $\frac{x}{2} =$ one-half of the number.

From the conditions of the problem,

$$\frac{x}{3}+\frac{x}{4}=\frac{x}{2}-\frac{x}{3}+50$$
$$4x+3x=6x-4x+600$$
$$5x=600$$
$$x=120, \text{ the required number.}$$

7. A can do a piece of work in 6 days, and B can do it in 8 days. In how many days can both do the work?

Let $x =$ the number of days required for both to do it.

$\frac{1}{6}=$ the part that A can do in 1 day,

and $\frac{x}{6}=$ the part that A can do in x days.

$\frac{1}{8}=$ the part that B can do in 1 day,

and $\frac{x}{8}=$ the part that B can do in x days.

Since they do *all* of the work in x days,

$$\frac{x}{6}+\frac{x}{8}=1$$
$$4x+3x=24$$
$$7x=24$$
$$x=3\tfrac{3}{7}, \text{ the number of days.}$$

8. A man is three times as old as his son, but in ten years he will be only twice as old. Find the age of each.

Let $x =$ the son's age now;

then $3x =$ the father's age now.

$x+10 =$ the son's age 10 years hence,

and $3x+10 =$ the father's age 10 years hence.

Since the father will be twice as old as the son 10 years hence,

$$\frac{3x+10}{2}=x+10$$
$$3x+10=2x+20$$
$$x=10, \text{ the son's age,}$$
$$3x=30, \text{ the father's age.}$$

9. At what time between 4 and 5 o'clock are the hands of a clock together?

Let $x =$ the number of minute-spaces passed over by the minute-hand before the hands are together;

then $\frac{x}{12} =$ the number of minute-spaces passed over by the hour-hand in the same time.

$$x = 20 + \frac{x}{12}$$

$$12x = 240 + x$$

$$11x = 240$$

$$x = 21\tfrac{9}{11}$$

The number of minute-spaces passed over by the minute-hand before the hands are together is $21\frac{9}{11}$, hence the time is $21\frac{9}{11}$ minutes after 4.

10. At what times between 5 and 6 o'clock are the hands of a clock at right angles to each other?

Let $x =$ the number of minute-spaces passed over by the minute-hand before the hands are at right angles to each other;

then $\frac{x}{12} =$ the number of minute-spaces passed over by the hour-hand in the same time.

It is evident that the hands are at right angles twice between 5 and 6, once before the minute-hand passes the hour-hand, and once after the minute-hand passes the hour-hand.

The two equations are as follows:

$$x = 25 + \frac{x}{12} - 15$$

$$x = 25 + \frac{x}{12} + 15$$

SIMULTANEOUS EQUATIONS.

237. An **Indeterminate Equation** is an equation containing two or more unknown quantities whose values can not be definitely determined.

$$x + y = 18 \qquad\qquad 2x - 2y = 14$$

238. It is evident that the values of the unknown quantities in an indeterminate equation can not be definitely found, for the value of either one is determined by the value assigned to the other. But if another equation is given, expressing different relations between the unknown quantities, there is but one pair of values that will satisfy both equations.

239. **Simultaneous Equations** are equations in which each unknown quantity represents the same value in the several equations.

$$x + y = 8 \qquad\qquad 3x - 2y = 16$$
$$x - y = 2 \qquad\qquad 2x + 3y = 28$$

240. **Independent Equations** are equations that express different relations between the unknown quantities, and neither can be reduced to the form of the other.

$$x + y = 12 \qquad\qquad 3x + 2y = 22$$
$$x - y = 10 \qquad\qquad 2x + 3y = 23$$

241. Every simple equation containing two unknown quantities is indeterminate, hence two independent simultaneous equations are necessary to determine the values of two unknown quantities.

242. To solve two simultaneous equations containing two unknown quantities, it is necessary to combine them in such a way as to form a single equation containing but one unknown quantity. This process is called *elimination.*

243. **Elimination** is the process of combining two simultaneous equations containing two or more unknown quantities in such a way as to obtain a single equation in which one of the unknown quantities does not appear.

Elimination by Addition or Subtraction.

244. Principles. — 1. *If equal quantities be added to equal quantities, the sums will be equal.*

2. *If equal quantities be subtracted from equal quantities, the remainders will be equal.*

245. Rule. — *Determine which of the two unknown quantities is to be eliminated.*

If necessary, multiply one or both equations by such a number as will make the coefficients of that unknown quantity the same in both equations.

If the signs of the quantity to be eliminated are unlike, add the equations, member to member; if the signs are alike, subtract one equation from the other, member from member.

EXAMPLES.

1.	$x + y = 8$	(1)
	$x - y = 6$	(2)
Adding (1) and (2),	$2x = 14$	(3)
Dividing (3) by 2,	$x = 7$	
Substituting in (1),	$7 + y = 8$	(4)
Transposing,	$y = 8 - 7$	(5)
	$y = 1$	

2	$3x + 3y = 9$	(1)
	$3x + 2y = 7$	(2)
Subtracting (2) from (1),	$y = 2$	
Substituting in (1),	$3x + 6 = 9$	(3)
Transposing,	$3x = 3$	(4)
Dividing (4) by 3,	$x = 1$	

3.

	$9x+4y=43$	(1)
	$3x+2y=17$	(2)
	$9x+4y=43$	(1)
Multiplying (2) by 2,	$6x+4y=34$	(3)
Subtracting (3) from (1),	$3x\quad=9$	(4)
Dividing (4) by 3,	$x=3$	
Substituting in (2),	$9+2y=17$	(5)
Transposing,	$2y=17-9$	(6)
	$2y=8$	(7)
Dividing (7) by 2,	$y=4$	

4.

	$3x+2y=21$	(1)
	$2x+3y=19$	(2)
Multiplying (1) by 2,	$6x+4y=42$	(3)
Multiplying (2) by 3,	$6x+9y=57$	(4)
Subtracting (3) from (4),	$5y=15$	(5)
Dividing (5) by 5,	$y=3$	
Substituting in (1),	$3x+6=21$	(6)
Transposing,	$3x=21-6$	(7)
	$3x=15$	(8)
Dividing (8) by 3,	$x=5$	

5.

	$ax+by=c$	(1)
	$bx-ay=d$	(2)
Multiplying (1) by a,	$a^2x+aby=ac$	(3)
Multiplying (2) by b,	$b^2x-aby=bd$	(4)
Adding (3) and (4),	$a^2x+b^2x=ac+bd$	(5)
	$(a^2+b^2)x=ac+bd$	(6)
Dividing (6) by a^2+b^2,	$x=\dfrac{ac+bd}{a^2+b^2}$	

To find the value of y, combine (1) and (2) and eliminate x.

NOTE 1. — Select the quantity to be eliminated which has the smaller coefficients, or which requires the smaller multipliers to make its coefficients equal.

NOTE 2. — If the coefficients of the quantity to be eliminated are prime to each other, each coefficient may be used as the multiplier for the other equation.

NOTE 3. — If the coefficients of the quantity to be eliminated are not prime to each other, multiply by such numbers as will produce their lowest common multiple.

ELIMINATION BY SUBSTITUTION.

246. RULE. — *Determine which of the two unknown quantities is to be eliminated.*

From either equation, find the value of that unknown quantity in terms of the other.

Substitute this value for the same unknown quantity in the other equation.

EXAMPLE.

$$3x + 2y = 8 \qquad (1)$$

$$5x - 3y = 7 \qquad (2)$$

Transposing $2y$ in (1), $3x = 8 - 2y$ (3)

Dividing (3) by 3, $x = \dfrac{8 - 2y}{3}$ (4)

Substituting in (2), $\dfrac{40 - 10y}{3} - 3y = 7$ (5)

Clearing of fractions, $40 - 10y - 9y = 21$ (6)

Transposing, $-10y - 9y = 21 - 40$ (7)

$-19y = -19$ (8)

Dividing (8) by -19, $y = 1$

Substituting in (2), $5x - 3 = 7$ (9)

Transposing, $5x = 7 + 3$ (10)

$5x = 10$ (11)

Dividing (11) by 5, $x = 2$

ELIMINATION BY COMPARISON.

247. PRINCIPLE. — *Things which are equal to the same thing are equal to each other.*

248. RULE. — *Determine which of the two unknown quantities is to be eliminated.*

From each equation, find the value of that unknown quantity in terms of the other.

Place these two values equal to each other.

EXAMPLE.

	$3x - 2y = 14$	(1)
	$4x + 3y = 30$	(2)
Transposing $-2y$ in (1),	$3x = 14 + 2y$	(3)
Transposing $3y$ in (2),	$4x = 30 - 3y$	(4)
Dividing (3) by 3,	$x = \frac{14 + 2y}{3}$	(5)
Dividing (4) by 4,	$x = \frac{30 - 3y}{4}$	(6)
By Art. 247,	$\frac{14 + 2y}{3} = \frac{30 - 3y}{4}$	(7)
Clearing of fractions,	$56 + 8y = 90 - 9y$	(8)
Transposing,	$8y + 9y = 90 - 56$	(9)
	$17y = 34$	(10)
Dividing (10) by 17,	$y = 2$	
Substituting in (1),	$3x - 4 = 14$	(11)
Transposing,	$3x = 14 + 4$	(12)
	$3x = 18$	(13)
Dividing (13) by 3,	$x = 6$	

249. Fractional simultaneous equations, the denominators of which are the unknown quantities and simple expressions, are readily solved without clearing of fractions.

EXAMPLES.

1. $$\frac{1}{x}+\frac{1}{y}=3 \qquad (1)$$

$$\frac{1}{x}-\frac{1}{y}=1 \qquad (2)$$

Adding (1) and (2), $$\frac{2}{x}\quad=4 \qquad (3)$$

Clearing of fractions, $$2=4x \qquad (4)$$

Transposing, $$-4x=-2 \qquad (5)$$

Dividing (5) by -4, $$x=\tfrac{1}{2}$$

Substituting in (1), $$2+\frac{1}{y}=3 \qquad (6)$$

Transposing, $$\frac{1}{y}=3-2 \qquad (7)$$

$$\frac{1}{y}=1 \qquad (8)$$

$$y=1$$

2. $$\frac{2}{x}+\frac{3}{y}=\frac{7}{3} \qquad (1)$$

$$\frac{3}{x}+\frac{2}{y}=\frac{9}{4} \qquad (2)$$

Multiplying (1) by 3, $$\frac{6}{x}+\frac{9}{y}=7 \qquad (3)$$

Multiplying (2) by 2, $$\frac{6}{x}+\frac{4}{y}=\frac{9}{2} \qquad (4)$$

Subtracting (4) from (3), $$\frac{5}{y}=\frac{5}{2} \qquad (5)$$

Clearing of fractions, $10 = 5y$ (6)

Dividing (6) by 5, $y = 2$

Substituting in (1), $\frac{2}{x} + \frac{3}{2} = \frac{7}{3}$ (7)

Transposing, $\frac{2}{x} = \frac{7}{3} - \frac{3}{2}$ (8)

$\frac{2}{x} = \frac{5}{6}$ (9)

Clearing of fractions, $12 = 5x$ (10)

Dividing (10) by 5, $x = 2\frac{2}{5}$

3. $\frac{3}{2x} + \frac{1}{3y} = \frac{53}{120}$ (1)

$\frac{5}{3x} - \frac{3}{2y} = \frac{7}{60}$ (2)

Multiplying (1) by $\frac{3}{2}$, $\frac{9}{4x} + \frac{3}{6y} = \frac{53}{80}$ (3)

Multiplying (2) by $\frac{1}{3}$, $\frac{5}{9x} - \frac{3}{6y} = \frac{7}{180}$ (4)

Adding (3) and (4), $\frac{101}{36x} = \frac{505}{720}$ (5)

Dividing (5) by 101, $\frac{1}{36x} = \frac{5}{720}$ (6)

Multiplying (6) by 36, $\frac{1}{x} = \frac{1}{4}$ (7)

Clearing of fractions, $x = 4$

Substituting in (2), $\frac{5}{12} - \frac{3}{2y} = \frac{7}{60}$ (8)

$-\frac{3}{2y} = \frac{7}{60} - \frac{5}{12}$ (9)

$$-\frac{3}{2y}=-\frac{18}{60} \qquad (10)$$

Dividing (10) by -3, $\frac{1}{2y}=\frac{1}{10}$ (11)

Clearing of fractions, $10=2y$ (12)

Dividing (12) by 2, $y=5$

4. $\frac{1}{x}-\frac{1}{y}=c$ (1)

$$\frac{a}{x}+\frac{b}{y}=d \qquad (2)$$

Multiplying (1) by a, $\frac{a}{x}-\frac{a}{y}=ac$ (3)

$$\frac{a}{x}+\frac{b}{y}=d \qquad (2)$$

Subtracting (3) from (2), $\frac{a}{y}+\frac{b}{y}=d-ac$ (4)

$$\frac{a+b}{y}=d-ac \qquad (5)$$

Clearing of fractions, $a+b=dy-acy$ (6)

Transposing, $dy-acy=a+b$ (7)

$$(d-ac)y=a+b \qquad (8)$$

Dividing (8) by $d-ac$, $y=\frac{a+b}{d-ac}$

Three or More Unknown Quantities.

250. To determine the values of two unknown quantities, two independent simultaneous equations are necessary. To determine the values of three unknown quantities, three independent simultaneous equations are necessary. And generally, there must be as many independent simultaneous equations as there are unknown quantities.

251. Groups of three or more simultaneous equations may be solved by either of the methods of elimination already given, but the method of elimination by addition or subtraction is generally the most convenient.

252. Rule. — *Determine which of the three unknown quantities is to be eliminated first.*

Combine one of the equations with each of the others, eliminating that unknown quantity each time. The two resulting equations will contain the same two unknown quantities.

Combine these two equations, eliminating one of the unknown quantities in them, and find the value of the other.

Substitute the value of that unknown quantity in one of the equations containing two unknown quantities, and find the value of the second unknown quantity.

Substitute the values of the two unknown quantities found in one of the original equations, and find the value of the third unknown quantity.

EXAMPLES.

1.

$$6x + 4y - 3z = 23 \qquad (1)$$
$$2x + 3y + 2z = 20 \qquad (2)$$
$$4x - 2y + 5z = 27 \qquad (3)$$

$$6x + 4y - 3z = 23 \qquad (1)$$

Multiplying (2) by 3, $$6x + 9y + 6z = 60 \qquad (4)$$

Subtracting (1) from (4), $$5y + 9z = 37 \qquad (5)$$

Multiplying (2) by 2, $$4x + 6y + 4z = 40 \qquad (6)$$

$$4x - 2y + 5z = 27 \qquad (3)$$

Subtracting (3) from (6), $$8y - z = 13 \qquad (7)$$

$$5y + 9z = 37 \qquad (5)$$

Multiplying (7) by 9, $$72y - 9z = 117 \qquad (8)$$

Adding (5) and (8), $$77y = 154 \qquad (9)$$

Dividing (9) by 77, $y = 2$

Substituting in (7), $16 - z = 13$ (10)

$-z = -3$ (11)

Dividing (11) by -1, $z = 3$

Substituting in (1), $6x + 8 - 9 = 23$ (12)

$6x = 24$ (13)

Dividing (13) by 6, $x = 4$

NOTE.—One or more of the equations may contain only two of the unknown quantities. Notice carefully the two following solutions.

2.

$$6x + 2y - 5z = 28 \quad (1)$$
$$5x - 3y + 2z = 17 \quad (2)$$
$$7x + 3z = 41 \quad (3)$$

Multiplying (1) by 3, $18x + 6y - 15z = 84$ (4)

Multiplying (2) by 2, $10x - 6y + 4z = 34$ (5)

Adding (4) and (5), $28x - 11z = 118$ (6)

Multiplying (3) by 4, $28x + 12z = 164$ (7)

Subtracting (6) from (7), $23z = 46$ (8)

Dividing (8) by 23, $z = 2$

Substituting in (3), $7x + 6 = 41$ (9)

$7x = 35$ (10)

Dividing (10) by 7, $x = 5$

Substituting in (1), $30 + 2y - 10 = 28$ (11)

$2y = 8$ (12)

Dividing (12) by 2, $y = 4$

3.

$$\begin{array}{lrr}
 & 3x + 2y = 28 & (1)\\
 & 8y - 3z = 19 & (2)\\
 & 6z - 4x = 18 & (3)\\
\hline
\text{Multiplying (1) by 4,} & 12x + 8y = 112 & (4)\\
 & -3z + 8y = 19 & (2)\\
\hline
\text{Subtracting (2) from (4),} & 12x + 3z = 93 & (5)\\
\text{Multiplying (3) by 3,} & -12x + 18z = 54 & (6)\\
\hline
\text{Adding (5) and (6),} & 21z = 147 & (7)\\
\text{Dividing (7) by 21,} & z = 7 & \\
\text{Substituting in (2),} & 8y - 21 = 19 & (8)\\
 & 8y = 40 & (9)\\
\text{Dividing (9) by 8,} & y = 5 & \\
\text{Substituting in (1),} & 3x + 10 = 28 & (10)\\
 & 3x = 18 & (11)\\
\text{Dividing (11) by 3,} & x = 6 &
\end{array}$$

PROBLEMS.

253. Many problems in algebra, which really contain two or more unknown quantities, can readily be solved by the use of a single equation, containing but one unknown quantity. This is possible with problems in which the relations between the unknown quantities are so simple that all of them can be expressed in terms of a single unknown quantity. When the relations between the unknown quantities are more complex, it is more convenient to introduce as many unknown quantities as there are unknown numbers. Such solutions will involve simultaneous equations. In all such problems, there must be enough conditions expressed so that as many independent equations can be formed as there are unknown numbers to be found.

1. The larger of two numbers exceeds three times the smaller by 14, and twice the larger exceeds five times the smaller by 36. Find the numbers.

Let $x =$ the larger number,
and $y =$ the smaller number.

$$x - 3y = 14$$
$$2x - 5y = 36$$

2. If 5 pounds of tea and 3 pounds of coffee cost $4.50, and 8 pounds of tea and 8 pounds of coffee at the same prices cost $8, what is the price of each per pound?

Let $x =$ the price of the tea in cents,
and $y =$ the price of the coffee in cents.

$$5x + 3y = 450$$
$$8x + 8y = 800$$

3. Find three numbers whose sum is 90. $\frac{1}{3}$ of the first plus $\frac{1}{5}$ of the second plus $\frac{1}{2}$ of the third is 28; and if 50 be added to the first, the sum will equal twice the second.

Let $x =$ the first number,
and $y =$ the second number,
and $z =$ the third number.

$$x + y + z = 90$$
$$\frac{x}{3} + \frac{y}{5} + \frac{z}{2} = 28$$
$$x + 50 = 2y$$

4. A certain number of three digits is equal to nine times the sum of the digits. The digit in tens' place is three times the digit in units' place; and if 97 be added to the number, the digits will be interchanged. Find the number.

Let $x =$ the digit in hundreds' place,
and $y =$ the digit in tens' place,
and $z =$ the digit in units' place.

$$100x + 10y + z = \text{the number.}$$
$$100x + 10y + z = 9(x + y + z)$$
$$y = 3z$$
$$100x + 10y + z + 97 = 100z + 10y + x$$

QUADRATIC EQUATIONS.

A Quadratic Equation is an equation of the second degree in the unknown quantity or quantities.

$3x^2 = 12.$ $\quad 3x^2 + a = x^2 - 9a.$ $\quad x^2 + 2xy = 15.$

A Pure Quadratic Equation is an equation which contains only the second power of the unknown quantity.

$5x^2 = 20.$ $\quad 2x^2 - c = 8c - x^2.$ $\quad x^2 - 2b = a.$

Solution of Pure Quadratic Equations.

$$15x^2 + 30 = 9x^2 + 180$$
$$6x^2 = 150$$
$$x^2 = 25$$
$$x = \pm 5$$

Rule.—*Reduce the equation to the form of $x^2 = a$, and extract the square root of both members.*

If the second member is not a square, indicate the square root of it by the use of the radical sign.

1. $x^2 + 8 = 16 - x^2.$
2. $5x^2 - 9 = 7 - 3x^2.$
3. $x^2 + 4 = 2x^2 - 5.$
4. $3x^2 - 10 = x^2 - 9.$
5. $x^2 - 14 = 18 - x^2.$
6. $4x^2 - 8 = x^2 + 10.$
7. $2x^2 + 5 = x^2 + 30.$
8. $x^2 + 18 = 2x^2 + 6.$
9. $(x + 1)^2 = 2x + 9.$
10. $3x^2 - 7a = x^2 + a.$
11. $5x^2 - 7 = 3x^2 + 29.$
12. $4x^2 - 23 = x^2 + 4.$
13. $7x^2 + 5 = 4x^2 + 32.$
14. $3x^2 + 4 = 2x^2 + 5.$

1. The square of a number exceeds the square of 3 fourths of the number by 175. Find the number.

$$2.\quad (x+3)^2 - 8 = 2(3x + 13).$$

3. The square of a number increased by the square of 1 half of it is 720. Find the number.

$$4.\quad \frac{2x^2-6}{3} + \frac{x^2+9}{5} - \frac{x^2+20}{2} = 3.$$

5. Two numbers are to each other as 4 to 7, and the sum of their squares is 585. Find the numbers.

$$6.\quad \frac{3x^2-3}{4} - \frac{5x^2+1}{9} - \frac{x^2-5}{5} = 0.$$

7. The product of two numbers is 528. The quotient of the greater divided by the less is $3\frac{2}{3}$. Find the numbers.

$$8.\quad \frac{2a}{x} + \frac{x}{a} = \frac{6a^2-x^2}{ax}. \qquad 9.\quad \frac{x+3}{x-3} + \frac{x-3}{x+3} = 3\tfrac{1}{3}.$$

10. The smaller of two numbers is $\frac{1}{4}$ of the larger, and the difference of their squares is 540. Find the numbers.

$$11.\quad \frac{13x^2}{3} - 15 = \frac{13x^2}{4} + x^2. \qquad 12.\quad \frac{3}{2-x} + \frac{3}{2+x} = 4\tfrac{1}{2}.$$

13. A rectangular field of six acres is $1\frac{2}{3}$ times as long as it is wide. Find the dimensions of the field.

$$14.\quad \frac{2a}{x-a} + \frac{5x+2a}{3x} = -1. \qquad 15.\quad \frac{2(x+6)}{3} = \frac{6x+x^2}{9}.$$

16. The sum of the squares of two consecutive numbers exceeds twice the smaller by 33. Find the numbers.

17. A man bought land for \$960, paying 3 fifths as many dollars per acre as there were acres in the piece. At what price per acre did he buy the land?

1. The product of three times and four times a certain number is 768. Find the number.

2. The product of the third and sixth parts of a certain number is 32. Find the number.

3. Find three numbers that are to each other as 2, 3, and 5, and the sum of whose squares is 608.

4. Two numbers are to each other as 3 to 8, and the difference of their squares is 1375. Find the numbers.

5. The sum of the squares of two consecutive numbers exceeds twice the larger by 71. Find the numbers.

6. There are 16 square yards in a piece of paper the length of which is 9 times the width. Find the length.

7. One number is $\frac{3}{5}$ of another, and the difference of their squares is 400. Find the numbers.

8. Five equal squares of paper contain 304 square inches less than a square 32 inches on each side. Find the length of each of the five squares.

9. The sum of the squares of two successive even numbers exceeds four times the smaller number by two hundred ninety-two. What are the numbers.

10. The dimensions of a rectangular field are to each other as 8 to 5, and the field contains twenty-five acres. Find the dimensions of the field in rods.

11. A man worked 12 times as many days as he received dollars per day and earned $147. How long did he work and what did he receive per day?

12. The area of one square field is four times that of another, and both together contain 1125 square rods. What is the length of a side of the smaller square?

RATIO.

397. **Ratio** is the relation of one number to another of the same kind, expressed by the quotient of the first divided by the second.

398. **The Terms** of a ratio are the two numbers compared.

399. **The Sign of Ratio** is a colon (:). By the definition of ratio, it must have the same signification as the sign of division.

400. Since a ratio is determined by division, a ratio may be indicated in three ways, as follows:

$$a \div b = \frac{a}{b} = a : b.$$

401. **The Antecedent** is the first term of the ratio.

402. **The Consequent** is the second term of the ratio.

403. **A Couplet** is the two terms of a ratio.

404. **A Ratio of Equality** is a ratio whose terms are equal.

405. **A Ratio of Greater Inequality** is a ratio whose first term is greater than the second.

406. **A Ratio of Less Inequality** is a ratio whose first term is less than the second.

407. **The Reciprocal** of a ratio is the ratio obtained by interchanging the terms. It is also called an *inverse ratio.*

408. **A Simple Ratio** is a ratio whose terms are expressed by single numbers.

409. **A Compound Ratio** is a combination of two or more simple ratios. It is compounded by taking the product of the corresponding terms of the simple ratios.

410. **A Duplicate Ratio** is the ratio of the squares of the two terms.

411. **A Triplicate Ratio** is the ratio of the cubes of the two terms.

412. **A Subduplicate Ratio** is the ratio of the square roots of the two terms.

413. **A Subtriplicate Ratio** is the ratio of the cube roots of the two terms.

414. Since any ratio may be expressed as a fraction, all the principles of fractions apply to ratios. For principles of fractions, see Art. 171. In stating them, it is only necessary to substitute the word *antecedent* for *numerator*, *consequent* for *denominator*, and *ratio* for the *value of the fraction*.

Simplify the following ratios:

1. $3\frac{2}{3} : 2\frac{3}{4}$.	2. 9 in. : 2 yd.	3. \$1.75 : \$2.
4. $8\frac{3}{4} : 1\frac{3}{5}$.	5. 4 rd. : 6 ft.	6. \$16 : \$.80.
7. $2\frac{5}{6} : 6\frac{4}{5}$.	8. 6 oz. : 3 lb.	9. \$4.20 : \$9.
10. $a^3b : ab^2$.	11. $4\sqrt{3} : 2\sqrt{2}$.	12. $12\sqrt{54} : \sqrt{2}$.
13. $xy^4 : x^2y$.	14. $2\sqrt[3]{3} : 3\sqrt{2}$.	15. $14\sqrt[3]{9} : 2\sqrt{3}$.

16. $x^2 - y^2 : x - y$. 17. $x^2 - 8x + 15 : x^2 - 5x$.

18. $x + y : x^2 - y^2$. 19. $x^2 + 7x + 12 : 3x + x^2$.

20. Arrange the ratios $3:4$, $5:6$, $\sqrt{25}:\sqrt{64}$, $2:3$, and $5\sqrt{40}:\sqrt{2}$ in descending order of magnitude.

21. What is the ratio compounded of $4:5$, $15:16$, and $12:6$? Of $14:18$, $15:7$, and $12:25$?

22. What is the ratio compounded of $a^3x : by^2$ and $b^2y^2 : a^2x$? Of $4a^2b : 3ab^2$, $3ax^3 : 2by^2$, and $b^2y^2 : 3ax^3$?

23. What is the ratio compounded of $32:15$ and the duplicate ratio of $5:4$?

24. What is the ratio compounded of $a^4x^4 : 2b^2y^2$ and the triplicate ratio of $2by : ax$?

25. What is the ratio compounded of $4 : 3$ and the subduplicate ratio of $3 : 8$?

26. What is the value of x when the ratio of $17 - x : 18 + x$ is two-thirds?

27. By what number must the terms of the ratio, $13 : 17$, be equally diminished to make it equal to the ratio of $5 : 8$?

PROPORTION.

415. **A Proportion** is the expression of the equality of two ratios. The equality may be indicated by the sign of equality or by the double colon, as follows:

$$a : b = c : d \qquad \frac{a}{b} = \frac{c}{d} \qquad a : b :: c : d$$

416. Four numbers are said to be in *proportion* when the ratio of the first to the second is equal to the ratio of the third to the fourth. The four numbers are called *proportionals*, and the fourth number is called a *fourth proportional*.

417. **The Terms** of a proportion are the four terms of the two equal ratios.

418. **The Extremes** of a proportion are the first and fourth terms.

419. **The Means** of a proportion are the second and third terms.

420. **A Continued Proportion** is a succession of two or more equal ratios, in which each consequent is the antecedent of the next ratio.

$$a : b :: b : c :: c : d :: d : e$$

421. Numbers are said to be in *continued proportion* when the first is to the second, as the second is to the third, as the third is to the fourth, and so on.

422. **A Mean Proportional** is the second of three numbers that are in continued proportion.

423. **A Third Proportional** is the third of three numbers that are in continued proportion.

Principles of Proportion.

424. The following principles of proportion are demonstrated by the use and transformations of equations. Only the various forms of the equations are given, and students are expected to determine how each form is obtained and what axioms are involved in the change.

425. *If four numbers are in proportion, the product of the extremes is equal to the product of the means.*

If
$$a : b = c : d$$
$$\frac{a}{b} = \frac{c}{d}$$
$$\therefore\ ad = bc$$

426. *A mean proportional between two numbers is the square root of their product.*

If
$$a : b = b : c$$
$$b^2 = ac$$
$$\therefore\ b = \sqrt{ac}$$

427. *Either extreme of a proportion is equal to the product of the means divided by the other extreme, and either mean is equal to the product of the extremes divided by the other mean.*

If
$$a : b = c : d$$
$$ad = bc$$
$$a = \frac{bc}{d} \qquad d = \frac{bc}{a} \qquad b = \frac{ad}{c} \qquad c = \frac{ad}{b}$$

428. *If the product of two numbers is equal to the product of two other numbers, the factors of either product may be made the extremes, and the factors of the other product the means, of a proportion.*

1.	2.	3.	4.
$ad = bc$	$ad = bc$	$ad = bc$	$ad = bc$
$\frac{ad}{bd} = \frac{bc}{bd}$	$\frac{ad}{cd} = \frac{bc}{cd}$	$\frac{ad}{ac} = \frac{bc}{ac}$	$\frac{ad}{ab} = \frac{bc}{ab}$
$\frac{a}{b} = \frac{c}{d}$	$\frac{a}{c} = \frac{b}{d}$	$\frac{d}{c} = \frac{b}{a}$	$\frac{d}{b} = \frac{c}{a}$
$\therefore\ a:b = c:d$	$\therefore\ a:c = b:d$	$\therefore\ d:c = b:a$	$\therefore\ d:b = c:a$

Obtain four proportions with a and d as the means.

429. *If four numbers are in proportion, they are in proportion by inversion; that is, the second term is to the first as the fourth is to the third.*

If
$$a:b = c:d$$
$$ad = bc$$
$$bc = ad$$
$$\therefore\ b:a = d:c$$

430. *If four numbers are in proportion, they are in proportion by alternation; that is, the first term is to the third as the second is to the fourth.*

If
$$a:b = c:d$$
$$ad = bc$$
$$\therefore\ a:c = b:d$$

431. *If four numbers are in proportion, they are in proportion by composition; that is, the sum of the terms of the first ratio is to either term as the sum of the terms of the second ratio is to the corresponding term.*

If $a:b=c:d$

$$\frac{a}{b}=\frac{c}{d}$$

$$\frac{a}{b}+1=\frac{c}{d}+1$$

$$\frac{a+b}{b}=\frac{c+d}{d}$$

$$\therefore\ a+b:b=c+d:d$$

If $b:a=d:c$

$$\frac{b}{a}=\frac{d}{c}$$

$$\frac{b}{a}+1=\frac{d}{c}+1$$

$$\frac{b+a}{a}=\frac{d+c}{c}$$

$$\therefore\ a+b:a=c+d:c$$

432. *If four numbers are in proportion they are in proportion by division; that is, the difference of the terms of the first ratio is to either term as the difference of the terms of the second ratio is to the corresponding term.*

If $a:b=c:d$

$$\frac{a}{b}=\frac{c}{d}$$

$$\frac{a}{b}-1=\frac{c}{d}-1$$

$$\frac{a-b}{b}=\frac{c-d}{d}$$

$$\therefore\ a-b:b=c-d:d$$

433. *If four numbers are in proportion, they are in proportion by composition and division; that is, the sum of the terms of the first ratio is to their difference as the sum of the terms of the second ratio is to their difference.*

If $a:b=c:d$

$$\frac{a+b}{b}=\frac{c+d}{d}$$

$$\frac{a-b}{b}=\frac{c-d}{d}$$

$$\frac{a+b}{a-b}=\frac{c+d}{c-d}$$

$$\therefore\ a+b:a-b=c+d:c-d$$

434. *In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

If $$a:b=c:d=e:f=g:h$$

$$ba=ab$$
$$bc=ad$$
$$be=af$$
$$bg=ah$$
$$(a+c+e+g)\,b=(b+d+f+h)\,a$$
$$\therefore\ a+c+e+g:b+d+f+h=a:b$$

Note. — Students should also be able to show that the sum of the antecedents is to the sum of the consequents as $c:d$, as $e:f$, and as $g:h$.

435. *The product of the corresponding terms of two or more proportions are in proportion.*

If $$a:b=c:d$$
$$e:f=g:h$$
$$\frac{a}{b}=\frac{c}{d}$$
$$\frac{e}{f}=\frac{g}{h}$$
$$\frac{ae}{bf}=\frac{cg}{dh}$$
$$\therefore\ ae:bf=cg:dh$$

436. *Like powers of proportionals are proportional.*

If $$a:b=c:d$$
$$\frac{a}{b}=\frac{c}{d}$$
$$\frac{a^n}{b^n}=\frac{c^n}{d^n}$$
$$\therefore\ a^n:b^n=c^n:d^n$$

437. *Like roots of proportionals are proportional.*

If $$a:b=c:d$$

$$\frac{a}{b}=\frac{c}{d}$$

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\frac{\sqrt[n]{c}}{\sqrt[n]{d}}$$

$$\therefore\ \sqrt[n]{a}:\sqrt[n]{b}=\sqrt[n]{c}:\sqrt[n]{d}$$

438. *If two proportions have a couplet in each the same, the other couplets will form a proportion.*

If $$a:b=c:d$$
$$a:b=e:f$$
$$\frac{a}{b}=\frac{c}{d}$$
$$\frac{a}{b}=\frac{e}{f}$$
$$\frac{c}{d}=\frac{e}{f}$$
$$\therefore\ c:d=e:f$$

If $$a:b=c:d$$
$$h:n=a:b$$
$$\frac{a}{b}=\frac{c}{d}$$
$$\frac{h}{n}=\frac{a}{b}$$
$$\frac{h}{n}=\frac{c}{d}$$
$$\therefore\ h:n=c:d$$

439. *If the antecedents of two proportions are the same, the consequents are in proportion.*

$$a:b=c:d$$
$$a:e=c:h$$
$$a:c=b:d$$
$$a:c=e:h$$
$$\therefore\ b:d=e:h$$

440. Equimultiples of two numbers are the products obtained by multiplying each of them by the same number.

4 times 5 and 4 times 7 are equimultiples of 5 and 7.

441. *If the first two terms of a proportion be multiplied by any number, and the last two terms by any number, the four products will be in proportion.*

If
$$a : b = c : d$$
$$\frac{a}{b} = \frac{c}{d}$$
$$\frac{am}{bm} = \frac{cn}{dn}$$
$$\therefore\ am : bm = cn : dn$$

PROBLEMS.

Find the value of x in the following proportions:

1. $3\frac{1}{8} : .02 :: x : .012.$
2. $\frac{4}{9} : x = 4 : 4\frac{1}{2}.$
3. $15 : 5 :: \$48.75 : x.$
4. $\frac{3}{7} : 2\frac{1}{7} = x : 5.$
5. $\$8.20 : \$3.28 :: x : 12.$
6. $x : 12 = \frac{8}{9} : \frac{2}{3}.$
7. 4 yd. : 5 ft. : : $\$2.04 : x.$
8. $\frac{5}{7} : x = 4\frac{1}{6} : 3\frac{1}{2}.$
9. $5 : x :: \$12.05 : \$21.69.$
10. $4\frac{1}{2} : x = 1\frac{3}{8} : 1\frac{5}{6}.$
11. $\$8.75 : x ::$ 4 rd. : 11 yd.
12. $x : 5 = .7 : 1.4.$
13. $x : 30 :: \$14.28 : \$35.70.$
14. $9\frac{1}{2} : 19 = x : 1\frac{1}{4}.$
15. $x - a : x - b :: x - c : x - d.$
16. $x + 7 : x - 2 :: 2x - 2 : x - 3.$
17. $ax + m : bx + n :: ax + c : bx + d.$
18. $4x + a : 4x + c :: 3x - d : 3x - b.$
19. $(x+7)(x-6) : (x-5)(x-6) :: (x+5)(x-3) : (x-2)(x-6).$

Find the mean proportional between

20. 16 and 4.
21. a^4 and a^2.
22. $12a^3x^2$ and $3a^3$.
23. 16 and 9.
24. a^3 and a^2.
25. $27a^2x^3$ and $3a^3$.
26. 25 and 2.
27. a^6 and x^4.
28. $18a^4x^2$ and $6x^3$.

Find the third proportional to

29. 12 and 6. **30.** a^4 and a^3. **31.** $48a^2$ and $8a^3x^3$.

32. 15 and $\sqrt{3}$. **33.** a^5 and a^3. **34.** $24x^3$ and $6a^2x^2$.

35. 14 and $\sqrt{7}$. **36.** a^2 and a^4. **37.** $30a^4$ and $5a^2x^3$.

Find the fourth proportional to

38. 8, 6, and 4. **39.** $2a^2$, $3a^3b$, and $4b^2$.

40. 7, 5, and 6. **41.** $3a^3$, $2ab^3$, and $5a^2$.

42. 4, 9, and 7. **43.** $5a^4$, $3a^3b$, and $6a^2$.

44. 3, 7, and 9. **45.** $4x^2$, $5b^2x$, and $8x^2$.

46. 9, 4, and 7. **47.** $7a^4$, $3a^2x$, and $8a^3$.

48. If $a : b :: c : d$, show that $ac : bd = c^2 : d^2$.

49. The sum of two numbers is 15, and the ratio of their squares is $2\frac{1}{4}$. Find the numbers.

50. The ratio of two numbers is $\frac{2}{3}$, and the sum of their squares is 468. Find the numbers.

51. If $a : b :: c : d$, show that $ab : cd = a^2 : c^2$.

52. The sides of a triangle are as $2 : 3 : 4$, and the perimeter is 135 feet. Find the sides.

53. The area of a rectangular field is 4 acres, and the ratio of the dimensions is $1\frac{3}{5}$. Find the dimensions.

54. Divide $64 between two men so that their shares will be to each other as 3 to 5.

55. There are two numbers which are to each other as 5 to 3; and if 5 be added to each number, the ratio of the resulting numbers will be $1\frac{1}{2}$. Find the numbers.

56. The ratio of the areas of two squares, whose sides differ by 20 rods, is $2\frac{1}{4}$. Find the side of each.

57. Eight years ago A's age was to B's as 3 to 2, but the ratio of their ages now is $1\frac{3}{8}$. Find their ages now.